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Replica Symmetry Breaking Condition Exposed by Random Matrix Calculation of Landscape Complexity

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Abstract We start with a rather detailed, general discussion of recent results of the replica approach to statistical mechanics of a single classical particle placed in a random $N(\gg 1)$-dimensional Gaussian landscape and confined by a spherically symmetric potential suitably growing at infinity. Then we employ random matrix methods to calculate the density of stationary points, as well as minima, of the associated energy surface. This is used to show that for a generic smooth, concave confining potentials the condition of the zero-temperature replica symmetry breaking coincides with one signaling that *both* mean total number of stationary points in the energy landscape, *and* the mean number of minima are exponential in N. For such systems the (annealed) complexity of minima vanishes cubically when approaching the critical confinement, whereas the cumulative annealed complexity vanishes quadratically. Different behaviour reported in our earlier short communication (Fyodorov et al. in JETP Lett. 85:261, 2007) was due to non-analyticity of the hard-wall confinement potential. Finally, for the simplest case of parabolic confinement we investigate how the complexity depends on the index of stationary points. In particular, we show that in the vicinity of critical confinement the saddle-points with a positive annealed complexity must be close to minima, as they must have a vanishing *fraction* of negative eigenvalues in the Hessian.

1 Introduction and General Discussion

After more than three decades of intensive research, understanding statistical mechanics of disordered systems still remains a considerable challenge to both theoretical and mathematical physics communities. In the situations where an interplay between thermal fluctuations and those due to quenched disorder is essential, one of the central problems is to find the averaged value of the free energy of the system as a function of temperature T. Namely, given an energy function which assigns a random value $\mathcal{H}(\mathbf{x})$ to every point $\mathbf{x} \in \Sigma$ of a

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given configuration space Σ, the task is to calculate

$$F = -T\langle \ln Z_\beta \rangle, \quad Z_\beta = \int_\Sigma \exp -\beta \mathcal{H}(\mathbf{x})\, d\mathbf{x}. \tag{1}$$

Here and henceforth the angular brackets stand for the disorder averaging, $\beta = 1/T$ is the inverse temperature, and Z_β is known as the partition function. In the case when the configuration space consists of discrete points (e.g. the hypercube $\Sigma = \{-1, 1\}^N$) the integration is replaced by an appropriate summation.

Confronted with the problem of performing such a calculation explicitly, theoretical physicists almost invariably choose to employ the so-called replica trick—a powerful, yet heuristic way of extracting the above average from the positive integer moments of the partition function. The trick amounts to exploitation of the formal identity

$$\langle \ln Z_\beta \rangle = \lim_{n\to 0} \frac{1}{n} \ln \langle Z_\beta^n \rangle, \quad Z^n = \int_{\mathbb{R}^N} e^{-\beta \sum_{a=1}^n \mathcal{H}(\mathbf{x}_a)} \prod_{a=1}^n d\mathbf{x}_a. \tag{2}$$

Indeed, assuming the distribution of random energies $\mathcal{H}(\mathbf{x})$ to be Gaussian with the prescribed covariance in the configuration space, the moments can be readily calculated for all values $n = 1, 2, \ldots$. Unfortunately, for all systems of interest those moments grow too fast with n to allow a unique restoration of the probability distribution of the partition function, hence prohibiting the honest evaluation of the expectation value of the logarithm. Yet, physicists managed to employ their fine intuition and developed a refined and efficient heuristic machinery of performing a mathematically ill-defined continuation $n \to 0$. In this way they were able to reconstruct highly nontrivial patterns of behaviour typical for disordered systems, the most intricate scenario being based on the notion of *Replica Symmetry Breaking* (RSB). According to that scenario first developed by Parisi in the context of the so-called mean-field models of spin glasses, see [1] as the standard reference and [2] for a recent account, below a certain transition temperature (frequently referred to as the de-Almeida-Thouless [3] temperature) the Boltzmann-Gibbs measure on the configuration space is effectively decomposed into a multitude of the so-called "equilibrium states" (this phenomenon is frequently referred to as an *ergodicity breaking*). The number of those states is argued to grow exponentially with the number of effective degrees of freedom, and they are believed to be organized in the configuration space in an intricate hierarchical way. Essentially, the structure is thought to represent a picture of valleys within valleys within valleys, etc., in an effective random (free) energy landscape [4] arising in such models. As that unusual picture arose via the application of a somewhat problematic replica trick, it is important to mention that a few rather nontrivial results emerging in the framework of the Parisi scenario were recently recovered by rigorous mathematical methods. These important developments are mainly due to recent seminal works by Talagrand [5, 6], based on earlier results by Guerra [7], see also interesting works by Aizenman, Sims and Starr [8]. In particular, Talagrand was able to demonstrate that the equilibrium free energy emerging naturally in the Parisi scheme of RSB is indeed the correct thermodynamic limit of the free energy, both for the paradigmatic Sherrington-Kirkpatrick model [6], and for the so-called spherical model of spin glasses [5].

One of the simplest, yet nontrivial representatives of a disordered system is arguably a single classical particle placed in a superposition of a random Gaussian $V(\mathbf{x})$ and a deterministic confining potential $V_{con}(\mathbf{x})$, with $\mathbf{x} \in \Sigma = \mathbb{R}^N$. It turned out to be a surprisingly rich model, characterised by a non-trivial dynamical behaviour as well as interesting thermodynamics. Works by Mezard and Parisi [9–11], and Engel [12] used the replica

trick to calculate the free energy of such a system with the simplest, parabolic confinement $V_{con}(\mathbf{x}) = \frac{1}{2}\mu\mathbf{x}^2$, $\mu > 0$ and after further employing the so-called Gaussian Variational Ansatz (GVA) revealed the existence of a low-temperature phase with broken ergodicity. They were followed by Franz and Mezard [13] and Cugliandolo and Le Doussal [14] papers on the corresponding dynamics revealing long-time relaxation, aging, and other effects typical for glassy type of behaviour at low enough temperatures. The nature of the low-temperature phase was found to be very essentially dependent on the type of correlations in the random potential, specified via the covariance function chosen in the form ensuring stationarity and well-defined large-N limit as

$$\left\langle V(\mathbf{x}_1)V(\mathbf{x}_2)\right\rangle = Nf\left(\frac{1}{2N}(\mathbf{x}_1-\mathbf{x}_2)^2\right), \tag{3}$$

with brackets standing for the averaging over the Gaussian potential distribution. Namely, if the covariance $f(x)$ decayed to zero at large arguments, the description of the low temperature phase was found to require only the so-called one-step replica symmetry breaking (1RSB) Parisi pattern. In contrast, for the case of long-ranged correlated potentials with $f(x)$ growing with x as a power-law[1] the full infinite-hierarchy Parisi scheme of replica symmetry breaking (FRSB) had to be used instead.

Based on formal analogies with the Hartree-Fock method Mezard and Parisi [9–11] argued that GVA-based calculations should become exact in the limit of infinite spatial dimension N. In a recent paper [15] the replicated problem was reconsidered in much detail by an alternative method which directly exposed the degrees of freedom relevant in the limit $N \to \infty$, and employed the Laplace (a.k.a. saddle-point) evaluation of the integrals. The method allowed also to perform calculations for any fixed sample size $|\mathbf{x}|^2 < L = R\sqrt{N}$. The effective radius $R < \infty$ can be used as an additional control parameter, and the results in the limit $R \to \infty$ and fixed $\mu > 0$ indeed reproduced those obtained by GVA in [9, 12]. In addition, the research in [15] and the subsequent work [16] revealed the existence of a non-trivial class of random potentials overlooked in earlier papers [9, 12], the simplest representative being the case of logarithmically growing correlations $f(x) = f_0 - g^2 \ln(x + a^2)$, with f_0, g, a being some positive constants. Indeed, for such a choice (and its generalizations [16]) the resulting free energy in the thermodynamic limit $R \to \infty$ turns out to coincide precisely with that appearing in the celebrated Generalized Random Energy Model (GREM) by Derrida [17–19], which is, in particular, known to describe directed polymers on a tree with disordered potentials [20]. Moreover, by comparing the results of [16] with earlier renormalization-group analysis of same system for finite dimensions $1 \le N < \infty$, see Carpentier and Le Doussal [21], it was conjectured that essentially the same type of behavior should survive in finite-dimensional systems. Note, that understanding the behaviour of the model for finite N in full generality remains a challenging problem, see e.g. [23] for a recent discussion.

According to a general wisdom, many peculiar features of systems with quenched disorder as compared to their more ordered counterparts are largely due to a multitude of nearly degenerate (metastable) states making the associated (free) energy landscape extremely corrugated, see e.g an insightful paper [22] trying to understand the gross structure of the energy landscapes describing elastic manifolds in presence of a random pinning potential. In particular, for the paradigmatic mean-field models of spin glasses a considerable effort has

[1] To be more precise, at large separations $\langle[V(\mathbf{x}_1) - V(\mathbf{x}_2)]^2\rangle \propto (\mathbf{x}_1 - \mathbf{x}_2)^{2\gamma}$, $0 < \gamma < 1$. However one can easily satisfy oneself that the difference is immaterial for the free-energy calculations.

gone into trying to understand what changes the intricate scenario of the replica symmetry breaking interpreted as ergodicity breaking (see our earlier discussion of the Parisi scenario) implies in the statistics of the associated free energy landscape (the so-called Thouless-Anderson-Palmer (TAP) [4] variational functional). For an instructive review of the recent efforts in this direction a reader may wish to look through [24], or to read directly representative original works on this topic [25–30].

For the model of interest for us in this paper—a single particle in a random Gaussian potential—the necessity (and difficulty) of adequately taking into account multiple minima in the energy landscape is well-known since the early work of Engel [31] who pointed out a non-perturbative character of the problem. In fact, similar non-perturbative effects are believed to invalidate the famous "dimensional reduction" phenomenon in the random-field Ising model, see [32, 33] and references therein. It is therefore important to develop methods and tools of statistical characterization of relevant features of N-dimensional random energy landscapes. Ideally, one would like to obtain a detailed knowledge on typical positions and depths of various minima and their mutual correlations. In such a general formulation the problem looks extremely challenging mathematically, although in the dimension $N = 1$ considerable progress can be achieved [34].

Obviously, the first meaningful step in this direction consists of counting the expected number of stationary points of different types (minima, maxima and saddles of various indices). The simplest, yet already a non-trivial problem of this sort is to find the mean number $\langle \#_s \rangle$ of *all* stationary points, irrespective of their index. Assuming that the landscape is described by a sufficiently smooth random function $\mathcal{H}$ of N real variables $\mathbf{x} = (x_1, \ldots, x_N)$ the problem amounts to finding all solutions of the simultaneous stationarity conditions $\partial_k \mathcal{H} = 0$ for all $k = 1, \ldots, N$, with ∂_k standing for the partial derivative $\frac{\partial}{\partial x_k}$. The total number $\#_s(D)$ of its stationary points in any spatial domain D is then given by $\#_s(D) = \int_D \rho_s(\mathbf{x})\, d\mathbf{x}$, with $\rho_s(\mathbf{x})$ being the corresponding density of the stationary points. The ensemble-averaged value of such a density can be found according to the multidimensional analogue of the so-called Kac–Rice formula:

$$\langle \rho_s(\mathbf{x}) \rangle = \left\langle \left| \det(\partial^2_{k_1,k_2} \mathcal{H}) \right| \prod_{k=1}^{N} \delta(\partial_k \mathcal{H}) \right\rangle, \tag{4}$$

where $\delta(x)$ stands for the Dirac δ-function. For the original Kac and Rice papers in $N = 1$ see [36, 37], the multidimensional generalization can be found e.g. in [38] and [39], and [40] contains further extensions and a proof in general setting. Note importance of keeping the modulus in (4), as omitting it would yield instead of the density the object related to the Euler characteristics of the surface, see [41] and discussions in [40] (see also the related identity (10) below).

Similarly, if one is interested in counting only minima, the corresponding mean density can be written as

$$\langle \rho_m(\mathbf{x}) \rangle = \left\langle \det(\partial^2_{k_1,k_2} \mathcal{H}) \theta(\partial^2_{k_1,k_2} \mathcal{H}) \prod_{k=1}^{N} \delta(\partial_k \mathcal{H}) \right\rangle. \tag{5}$$

Here and henceforth we use the notation $\theta(x)$ for the Heaviside step-function, i.e. $\theta(x) = 1$ for $x > 0$, and zero otherwise. The corresponding matrix θ-factor in (5) selects only stationary points with positive definite Hessians, which are minima.

In the context of the problematic of glassy systems calculation of those quantities for the simplest case $N = 1$ were presented in [35]. Actually, low-dimensional cases $N = (1, 2, 3)$

were known much earlier, starting from the classical papers by Longuet-Higgins on specular light reflection from a rough sea surface [42], and the work on laser speckle patterns by Weinrib and Halperin [43]. The case $N = 3$ was also considered in early work [44] in the context of behaviour of a *quantum* particle in a random potential, and recently in [45]. A useful summary of related efforts in various areas of applied mathematics and probability theory can be found in a recent book by Adler and Taylor [40].

Random landscapes relevant in statistical physics are however mainly high-dimensional. In particular, general interest in landscapes behind glass-forming mixtures of various types led to a few numerical investigations of the structure of their critical points, see e.g. [46–49]. Independently, interest in counting extrema of some random high-dimensional surfaces arose also in string theory where an effective string landscape is believed to possess a huge number of possible minima, each of which could describe a potential universe, see [50] for further discussion, references and actual calculations. At the same time, no analytical results on numbers of critical points were available for high-dimensional Gaussian landscapes until very recently, before it was realized that obtaining them requires exploitation of well-developed random-matrix based techniques [51]. For the N-dimensional Gaussian random surface with parabolic confinement, i.e. for the function

$$\mathcal{H}_{par}(\mathbf{x}) = \frac{\mu}{2}\mathbf{x}^2 + V(\mathbf{x}), \quad \mu > 0 \tag{6}$$

the total expected number of stationary points calculated from the Kac–Rice density (4) turned out to be given by [51]:

$$\langle \#_s \rangle = \frac{1}{\mu^N} \langle |\det(\mu \hat{I}_N + \hat{H})| \rangle, \tag{7}$$

where $\hat{H}$ stands for $N \times N$ Hessian matrix of second derivatives of the Gaussian part of the potential: $H_{ij} \equiv \partial^2_{ij} V(\mathbf{x})$ $\{(i, j) = 1, \ldots, N\}$, and $\hat{I}_N$ for the identity matrix, and we took into account that the distribution of the Hessian is position-independent due to translational invariance of the covariance function (3), see (8) below. We see that the problem basically amounts to evaluating ensemble average of the absolute value of the characteristic polynomial of a certain real symmetric random matrix, whose entries are centered Gaussian variables due to the nature of the underlying potential $V(\mathbf{x})$. The form of the covariance function (3) implies after a simple calculation the following covariances of the entries H_{ij}:

$$\langle H_{il} H_{jm} \rangle = \frac{f''(0)}{N} [\delta_{ij}\delta_{lm} + \delta_{im}\delta_{lj} + \delta_{il}\delta_{jm}], \tag{8}$$

where here and henceforth in the paper the number of dashes indicates the number of derivatives taken. This allows one to write down the density of the joint probability distribution (JPD) of the matrix H explicitly as [51]

$$\mathcal{P}(\hat{H}) d\hat{H} \propto d\hat{H} \exp\left\{ -\frac{N}{4f''(0)} \left[\mathrm{Tr}(\hat{H}^2) - \frac{1}{N+2} (\mathrm{Tr}\hat{H})^2 \right] \right\}, \tag{9}$$

where $d\hat{H} = \prod_{1 \le i \le j \le N} dH_{ij}$ and the proportionality constant can be easily found from the normalisation condition. Although such a JPD is invariant with respect to rotations $\hat{H} \to \hat{O}^{-1}\hat{H}\hat{O}$ by orthogonal matrices $\hat{O} \in O(N)$ it is apparently different from the standard one typical for the so-called Gaussian Orthogonal Ensemble (GOE) of random symmetric matrices (the standard introduction into the Random Matrix Theory is [52]). A peculiarity of

this ensemble is particularly manifest via the following identity that holds for an *arbitrary* fixed matrix $\hat{A}$ [45]:

$$\langle \det(\hat{A} + \hat{H}) \rangle = \det \hat{A}. \tag{10}$$

In particular, this identity gives another evidence for importance of keeping the modulus in formulae like (7), where omitting the modulus would trivially yield unity due to (10). An elegant general verification of (10) directly from (8) can be found in [45]. Alternatively, in Appendix 1 we demonstrate its validity for real symmetric matrices by a specific method exploiting the very meaning of the matrix $\hat{H}$ as the Hessian matrix, as the latter is related to topological properties of the underlying gradient field.

In spite of all the mentioned peculiarities, actual evaluation of the ensemble average in (7) can be reduced to the standard random matrix calculation of the GOE density and as such can be performed in a closed form for any N in terms of Hermitian polynomials [51]. This is important, as the same method actually proved to be useful in other similar problems, as e.g. the problem of determining the distribution of the maximum of a Gaussian random field, see [53].

Large N asymptotics of the mean number of stationary points can be extracted from known behavior of Hermite polynomials. The explicit calculation revealed [51] that the replica symmetry breaking (interpreted in the standard way as ergodicity breaking) in the zero-temperature limit of the $R = \infty$ version of the model with parabolic confinement is accompanied by the emergence of an exponentially large mean total number of stationary points in the energy landscape. As common wisdom was to expect that the low-temperature thermodynamics should be dominated by minima rather than by the totality of stationary points the issue called for further investigation, in particular in a general $R < \infty$ case. In two recent short communications [54, 55] the calculations of [51] were independently and essentially generalized in a way providing access to the densities of minima [55], as well as to stationary points with an arbitrary index [54], and at a given value of the potential, hence energy.

The results in [55] were given as a function of both parameters μ and R. That analysis revealed that generally the domain of existence of the glassy phase with broken ergodicity (at zero temperature T) turns out to be always associated with the existence of exponentially many stationary points in the energy landscape, but not necessarily exponentially many minima. In an attempt to extend those considerations to finite temperatures the authors also constructed a simple variational functional providing an upper bound on the true free energy of the $\mu > 0$, $R = \infty$ version of the problem. Surprisingly, counting stationary points in that simple-minded approximation was already enough to capture such a nontrivial feature as the precise position of the de-Almeida-Thouless line in the whole (μ, T) plane.

The aim of the present paper is to describe the random-matrix evaluation of landscape complexities in a rather general setting. The calculations are performed for a broad class of Gaussian landscapes $V(\mathbf{x})$ with monotonically increasing, concave confining potentials, i.e. for the energy functions

$$\mathcal{H}(\mathbf{x}) = NU\left(\frac{\mathbf{x}^2}{2N}\right) + V(\mathbf{x}), \tag{11}$$

assuming $U'(z) > 0, U''(z) \geq 0, \ \forall z \geq 0$, with the first (confining) term chosen in the rotationally-invariant scaling form ensuring a well-defined large N limit of the model. Solving this problem will also be used as an opportunity to provide a rather detailed exposition of the techniques sketched in our earlier short communication [55]. We will be able to treat

in considerable generality the smooth confinement case, and compare it with non-analytic (hard-wall) confinement studied in [55] (and independently by A. Bray and D. Dean in [54]).

Our main results can be briefly summarized as follows. Define the function

$$\mu(z) = U'\left(\frac{z}{2}\right). \tag{12}$$

We find that for a generic, smooth concave confining potentials $U(z)$ with a continuous derivative $\mu(z)$ the condition ensuring that the mean *total* number of stationary points in the energy landscape is exponentially large in N coincides with one ensuring that the same is true for the mean number of minima. Different behaviour of two complexities for a particle confined in a finite box reported in [54, 55] is shown to be a consequence of non-analyticity of the hard-wall confinement potential. For a smooth confinement the common condition of positivity of the corresponding landscape complexities (defined as the logarithms of the mean total number per degree of freedom N) reads

$$\mu\left(-\frac{f'(0)}{f''(0)}\right) < \sqrt{f''(0)}. \tag{13}$$

As shown in Appendix 4, such an inequality is precisely the condition of instability of the zero-temperature replica symmetric solution of the associated problem in statistical mechanics,[2] see (152). The corresponding annealed complexity of minima vanishes cubically when approaching the critical confinement (60), whereas the cumulative annealed complexity vanishes quadratically with the distance to criticality. Finally, in the Appendix 5 we further investigate, using the method due to Bray and Dean [54] the annealed complexities of stationary points at a fixed value of the *index* $\mathcal{I}$ (the number of the negative eigenvalues of the Hessian). The calculation is performed for the simplest case of parabolic confinement (6), when $\mu(z) = const = \mu$, hence (60) amounts to $\mu < \mu_{cr} = \sqrt{f''(0)}$. The results reveal that the only stationary points with non-vanishing annealed complexity arising precisely at the critical value $\mu = \mu_{cr}$ are those for which the number of negative directions is not extensive: $\lim_{N\to\infty} \frac{\mathcal{I}}{N} = 0$. As we move inside the glassy phase away from the critical value $\mu = \mu_{cr}$, stationary points with an increasing range of indices start to have a positive complexity.

In the Appendices 1, 2, and 3 we provide a few useful technical details, some of them are not immediate to find in the available literature. We comment on the geometric features of the landscapes behind the curious formula (10) in the Appendix 1, and discuss the distribution of the diagonal element of the resolvent of a random GOE matrix in the Appendix 2. In the Appendix 3 we provide a short overview of a powerful heuristic random-matrix technique due to Dean and Majumdar [59] based on the functional integration. We found this method indispensable when calculating the complexity of minima. Finally, a statistical-mechanics calculation used for extracting the point of zero-temperature replica symmetry breaking for the present class of models within the framework of the replica method is sketched in the Appendix 4, and the calculation of complexity of stationary points with a given (extensive) index is sketched in Appendix 5.

[2]This result, in particular, gives a kind of *a posteriori* justification of the use of the mean value of stationary points/extrema as sensible characteristics of high-dimensional landscapes. Indeed, in the ideal world we would like to know the typical rather than the mean values, i.e. to replace "annealed" averages featuring in (4) and (5) with the corresponding "quenched" ones. The latter would require performing the averaging of the *logarithm* of the number of stationary points. Unfortunately, the present level of art in the field makes such a quenched calculation hardly feasible.

2 Complexity of Stationary Points Versus Complexity of Minima

2.1 The Density of Stationary Points for Spherically-Symmetric Potentials

Our goal is to study the density of stationary points of the function (11) around position $\mathbf{x}$, subject to the condition that the random part $V(\mathbf{x})$ takes the prescribed value V. This is given by a variant of the Kac–Rice formula (4):

$$\left\langle \rho_s(V,\mathbf{x},[U])\right\rangle = \left\langle \left|\det\left[\frac{\partial^2\mathcal{H}}{\partial\mathbf{x}\partial\mathbf{x}}\right]\right|\delta\left(\frac{\partial\mathcal{H}(\mathbf{x})}{\partial\mathbf{x}}\right)\delta[V-V(\mathbf{x})]\right\rangle, \tag{14}$$

where we used self-explanatory short-hand notations.

For the choice of the confining potential in (11) we obviously have for the gradient vector and for the Hessian matrix

$$\frac{\partial\mathcal{H}}{\partial\mathbf{x}} = \mathbf{x}\mu\left(\frac{\mathbf{x}^2}{N}\right) + \frac{\partial V}{\partial\mathbf{x}}, \tag{15}$$

$$\frac{\partial^2\mathcal{H}}{\partial\mathbf{x}\partial\mathbf{x}} = \mu\left(\frac{\mathbf{x}^2}{N}\right)\hat{I}_N + \hat{M}(\mathbf{x}) + \frac{\partial^2 V}{\partial\mathbf{x}\partial\mathbf{x}}, \tag{16}$$

where we used the function $\mu(z)$ defined according to (12), as well as $N\times N$ rank-one matrix $\hat{M}(\mathbf{x})$ with entries expressed via components x_i, $i=1,\ldots,N$ of $\mathbf{x}$ as

$$[\hat{M}(\mathbf{x})]_{ij} = \frac{x_i x_j}{N}U''\left(\frac{\mathbf{x}^2}{2N}\right) \equiv 2\frac{x_i x_j}{N}\mu'\left(\frac{\mathbf{x}^2}{N}\right). \tag{17}$$

Note that for the parabolic confinement case (6) obviously $\mu(z)\equiv\mu$. We assume in the subsequent analysis that $\mu(z)$ is non-decreasing: $\mu'(z)\geq 0, \forall z\geq 0$.

We find it also convenient to collect all disorder-dependent factors in the function

$$\mathcal{F}_s(V,\mathbf{x},\hat{K},[U]) = \left\langle \delta[V-V(\mathbf{x})]\delta\left[\mathbf{x}\mu\left(\frac{\mathbf{x}^2}{N}\right)+\frac{\partial V}{\partial\mathbf{x}}\right]\delta\left[\hat{K}-\frac{\partial^2 V}{\partial\mathbf{x}\partial\mathbf{x}}\right]\right\rangle \tag{18}$$

and use the above expressions to rewrite the density (14) in the form

$$\left\langle \rho_s(V,\mathbf{x},[U])\right\rangle = \int d\hat{K}\left|\det\left[\hat{K}+\hat{I}_N\mu\left(\frac{\mathbf{x}^2}{N}\right)+\hat{M}(\mathbf{x})\right]\right|\mathcal{F}(V,\mathbf{x},\hat{K},[U]). \tag{19}$$

In the rest of the calculations we assume that the covariance function $f(x)$ in (3) has finite values of the first two derivatives at the origin: $|f'(0)|<\infty$ and $|f''(0)|<\infty$. Actually, from the point of view of statistical mechanics such a condition ensures the existence of a nontrivial phase transition at zero temperature, see (13). Note that in the case of long-ranged potentials behaving like $f(x)\sim x^\gamma$ with $0<\gamma<1$ at large x this condition requires imposing some regularization at small arguments, e.g. $f(x)=(a+x)^\gamma$, $a>0$.

Let us introduce the following notations for the subsequent use, remembering $f'(0)<0$:

$$\mu_{cr} = \sqrt{f''(0)}, \quad R_{cr} = \sqrt{|f'(0)|/f''(0)}, \quad g^2 = f''(0) - \frac{f'(0)^2}{f(0)} \geq 0. \tag{20}$$

Our analysis of (18) starts with introducing the Fourier integral representation for each of the delta-functional measures—of scalar, vector, and matrix argument, correspondingly.

This step facilitates performing the ensemble average explicitly as averaging exponentials containing terms linear with respect to Gaussian variables requires only the knowledge of their covariances. In the course of this procedure one has to exploit a few identities which follow from (3) after simple calculations, as well as from the expressions (8). Namely,

$$\langle V^2(\mathbf{x})\rangle = Nf(0),\ \left\langle \frac{\partial V}{\partial \mathbf{x}}\frac{\partial V}{\partial \mathbf{x}}\right\rangle = -f'(0),\quad \left\langle V\frac{\partial V}{\partial \mathbf{x}}\right\rangle = \left\langle \frac{\partial V}{\partial \mathbf{x}}\frac{\partial^2 V}{\partial \mathbf{x}\partial \mathbf{x}}\right\rangle = 0, \tag{21}$$

$$\left\langle \left[\mathrm{Tr}\left(\hat{A}\frac{\partial^2 V}{\partial \mathbf{x}\partial \mathbf{x}}\right)\right]^2\right\rangle = \frac{1}{N}f''(0)[2\mathrm{Tr}\hat{A}^2 + (\mathrm{Tr}\hat{A})^2], \tag{22}$$

$$\left\langle V(\mathbf{x})\mathrm{Tr}\left(\hat{A}\frac{\partial^2 V}{\partial \mathbf{x}\partial \mathbf{x}}\right)\right\rangle = f'(0)\,\mathrm{Tr}\hat{A},\quad \left\langle \left(\mathbf{a}\frac{\partial V}{\partial \mathbf{x}}\right)^2\right\rangle = -f'(0)\mathbf{a}^2, \tag{23}$$

where $\mathbf{a}$ and $\hat{A}$ are an arbitrary N-component vector and an $N\times N$ real symmetric matrix, respectively.

After these steps one can easily integrate out the scalar and vector Fourier variables as they appear as simple quadratic terms in the exponential (the integration over the matrix Fourier variable requires a little bit more care, and is done at the next step). Thus, we arrive at the following expression:

$$\mathcal{F}(V,\mathbf{x},\hat{K},[U]) \propto \exp\left\{\frac{\mathbf{x}^2}{2f'(0)}\left[\mu\left(\frac{\mathbf{x}^2}{N}\right)\right]^2 - \frac{V^2}{2Nf(0)}\right\}\mathcal{I}(\hat{K}) \tag{24}$$

where $\mathcal{I}(\hat{K})$ stands for the integral over the remaining Fourier variables, that is over a real symmetric $N\times N$ matrix $\hat{P}$, and is given by

$$\mathcal{I}(\hat{K}) = \int \exp\left\{-\frac{\mu_{cr}^2}{N}\mathrm{Tr}\hat{P}^2 + i\mathrm{Tr}\hat{P}\left(\hat{K} - \frac{f'(0)}{Nf(0)}V\hat{I}_N\right) - \frac{g^2}{2N}(\mathrm{Tr}\hat{P})^2\right\}d\hat{P}. \tag{25}$$

Here, and henceforth, we will systematically disregard various multiplicative constant factors for the sake of brevity. They can always be restored from the normalisation conditions whenever necessary.

To deal with the remaining integral over the matrix $\hat{P}$ in (25) in the most economic way it is convenient to employ first a Gaussian integral over an auxiliary scalar variable t, and in this way to "linearize" the last term in the exponential in (25):

$$e^{-\frac{g^2}{2N}(\mathrm{Tr}\hat{P})^2} = \left(\frac{N}{2\pi}\right)^{1/2}\int e^{-\frac{N}{2}t^2 \pm igt\,\mathrm{Tr}\hat{P}}dt. \tag{26}$$

Substituting this identity back into (25) converts the matrix integral into a standard Gaussian one which can be immediately performed, yielding

$$\mathcal{F}(V,\mathbf{x},\hat{K},[U]) \propto \exp\left\{\frac{\mathbf{x}^2}{2f'(0)}\left[\mu\left(\frac{\mathbf{x}^2}{N}\right)\right]^2 - \frac{V^2}{2Nf(0)}\right\} \times \int_{-\infty}^{\infty} e^{-\frac{N}{2}t^2 - \frac{N}{4\mu_{cr}^2}\mathrm{Tr}[\hat{K} - (gt + \frac{f'(0)}{f(0)}\frac{V}{N})\hat{I}_N]^2}dt. \tag{27}$$

Before substituting this expression back into (19), we find it convenient to rescale coordinates and the value of the potential as $\mathbf{x}\to\sqrt{N}\mathbf{x}$ and $V\to NV$, and to introduce two

matrices $\hat{H}$ and $\hat{\mathcal{M}}$ defined as

$$\hat{H} = \hat{K} - \left(gt + \frac{f'(0)}{f(0)} V \right) \hat{I}_N \tag{28}$$

and

$$\hat{\mathcal{M}} = s\hat{I}_N + 2\frac{\mu'(\mathbf{x}^2)}{\mu_{cr}} \mathbf{x} \otimes \mathbf{x}^T, \tag{29}$$

where we introduced a short-hand notation

$$s = \frac{1}{\mu_{cr}} \left[\mu(\mathbf{x}^2) + gt + \frac{f'(0)}{f(0)} V \right] \tag{30}$$

and the diadic product $\mathbf{x} \otimes \mathbf{x}^T$ stands for the (rank-one) $N \times N$ matrix with components $x_i x_j$.

As a result, we find that the mean density of stationary points (19) is expressed in terms the rescaled variables as

$$\rho(V, \mathbf{x}, [U]) = \mathcal{N}_s \exp N \left(\frac{\mathbf{x}^2}{2f'(0)} \mu^2(\mathbf{x}^2) - \frac{V^2}{2f(0)} \right) \int dt e^{-Nt^2/2} \mathcal{D}_s(t, V, [U]), \tag{31}$$

$$\mathcal{D}_s(t, V, [U]) = \langle |\det(\hat{H} + \hat{\mathcal{M}})| \rangle_{GOE}, \tag{32}$$

where $\mathcal{N}_s$ is the required normalisation constant and $\langle \ldots \rangle_{GOE}$ denotes the average with respect to the standard GOE measure $\mathcal{P}(\hat{H}) d\hat{H} \propto \exp(-N \operatorname{Tr} \hat{H}^2/4)\, d\hat{H}$. Now we can use the rank-one character of the second term in (29) which implies the following identity

$$\det(\hat{H} + \hat{\mathcal{M}}) = \left[1 + 2\frac{\mu'(\mathbf{x}^2)}{\mu_{cr}} G_H(\mathbf{x}) \right] \det(\hat{H} + s\hat{I}), \tag{33}$$

where

$$G_H(\mathbf{x}) = \mathbf{x}^T \frac{1}{\hat{H} + s\,\hat{I}} \mathbf{x} \tag{34}$$

is a diagonal element of the resolvent of the random matrix $\hat{H} + s\,\hat{I}$. Note that with the parameter s being real as required by the identity (33), the resolvent can take arbitrary large values when an eigenvalue of the random matrix H occurs close enough to $-s$. As in the large N limit the eigenvalues of GOE matrices fill in densely the interval $(-2, 2)$ [52] we can expect the resolvent to retain strong fluctuations for $|s| < 2$. Outside that interval however we may expect self-averaging of the resolvent. The explicit calculation performed in the Appendix 2 confirms this intuition, and establishes the magnitude of the fluctuations inside the spectrum. Namely, we show that for $N \to \infty$ the distribution function of $G_H(\mathbf{x})$ tends to the Cauchy law centered around the mean value $\langle G_H(\mathbf{x}) \rangle_{GOE} = \frac{1}{2} s\, \mathbf{x}^2$ and with the widths $\Gamma = \langle |G_H(\mathbf{x}) - \langle G_H(\mathbf{x}) \rangle_{GOE}| \rangle_{GOE} = \frac{1}{2}\sqrt{4 - s^2}$ for $|s| < 2$, and $\Gamma = 0$ otherwise. We therefore conclude that the first factor in (33) is typically of the order of unity at every realisation of the disorder. Recalling that only factors of order $\exp\{O(N)\}$ in the mean density of stationary points are relevant for calculating the complexity, we can safely disregard the mentioned factor in the rest of the calculation by replacing (32) with

$$\mathcal{D}_s(t, V, [U]) = \langle |\det(\hat{H} + s\hat{I})| \rangle_{GOE} \tag{35}$$

and remembering the relation (30) of s to parameters of the problem.

For the asymptotic analysis of complexity we further use the result proved in [51]. It essentially claims that the expectation of the determinant in the right-hand side of (35) can be represented in the large N limit as

$$\mathcal{D}_s(t, V, [U]) \propto \exp N\Phi(s) \tag{36}$$

where the function $\Phi(s) = \Phi(-s)$ is given explicitly by

$$\Phi(s \geq 0) = \frac{s^2}{4} - \theta(s-2)\left[\frac{s\sqrt{s^2-4}}{4} + \ln\left(\frac{s-\sqrt{s^2-4}}{2}\right)\right]. \tag{37}$$

The method used in [51] was based on relating $\mathcal{D}_s(t, V, [U])$ to the mean eigenvalue density of GOE, and subsequent lengthy saddle-point analysis of an integral representation for that density. Perhaps, a shorter way to understand the above asymptotics is to notice that actually,

$$\Phi(s) = \int_{-2}^{2} \ln|s+\lambda|\rho_{sc}(\lambda)\,d\lambda, \quad \rho_{sc}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \tag{38}$$

where the integral is understood in the sense of the principal value. The above formula can be verified by applying ideas from statistical mechanics to the evaluation of $D_s(t, V)$. A mathematically rigorous method of analysing this type of problems can be found in a detailed paper by Boutet de Monvel, Pastur and Shcherbina [58]. An alternative, and very transparent way of heuristic asymptotic analysis based on the concept of a functional integral was proposed in a recent insightful work by Dean and Majumdar [59]. We provide an overview of the Dean-Majumdar approach in the Appendix 3.

We thus arrive to the following expression for the density of stationary points

$$\rho(V, \mathbf{x}, [U]) \sim \mathcal{N}_s \exp\left(N\left[\frac{\mathbf{x}^2}{2f'(0)}\mu^2(\mathbf{x}^2) - \frac{V^2}{2f(0)}\right]\right)\int_{-\infty}^{\infty} dt\, e^{N[\Phi(s)-t^2/2]} \tag{39}$$

valid in the large-N limit up to factors of order of unity.

2.2 The Cumulative Complexity of Stationary Points for a Smooth Confinement

The mean total number of the stationary points for a given shape of the spherically-symmetric confining potential is obtained by further integrating this density over the position $\mathbf{x}$ and over the values of the potential V:

$$\langle \#_s \rangle = \int_{\mathbb{R}^N} d\mathbf{x} \int_{-\infty}^{\infty} \rho(V, \mathbf{x}, [U])\, dV. \tag{40}$$

It is natural to use the fact that the integrand is spherically symmetric by passing to the integration over the radial variable $q = \mathbf{x}^2$ and the angular coordinates Ω, so that $d\mathbf{x} \propto q^{(N-2)/2} dq\, d\Omega$. As a result, we have

$$\langle \#_s \rangle \propto \int_0^{\infty} e^{N\mathcal{L}(q)} \frac{dq}{q} \mathcal{F}(q), \qquad \mathcal{F}(q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-N\mathcal{S}[V,q,t]}\, dt\, dV, \tag{41}$$

where

$$\mathcal{S}(V, q, t) = \frac{V^2}{2f(0)} + \frac{t^2}{2} - \Phi(s), \qquad \mathcal{L}(q) = \frac{q}{2f'(0)}\mu^2(q) + \frac{1}{2}\ln q, \tag{42}$$

with (cf. (30))

$$s = \frac{1}{\mu_{cr}}\left[\mu(q) + gt + \frac{f'(0)}{f(0)}V\right] \tag{43}$$

and $\Phi(s)$ given by (37). As usual, we disregarded in (41) all constant proportionality factors which will be restored at a later stage from the normalisation condition.

For our analysis of this expression in the limit $N \gg 1$ we find it convenient to evaluate first the integrals over t, V with the Laplace method. We find $\mathcal{F}(q) \propto \exp[-N\mathcal{S}(V_*, t_*)]$, with V_* and t_* being the values minimizing $\mathcal{S}(V, q, t)$ for a given q, and satisfying the system of equations:

$$V_* = \frac{f'(0)}{\mu_{cr}}\frac{d\Phi(s)}{ds}\bigg|_{s=s_*}, \qquad t_* = \frac{g}{\mu_{cr}}\frac{d\Phi(s)}{ds}\bigg|_{s=s_*}, \tag{44}$$

where according to (43)

$$s_* = \frac{1}{\mu_{cr}}\left[\mu(q) + gt_* + \frac{f'(0)}{f(0)}V_*\right] \equiv \frac{1}{\mu_{cr}}\left[\mu(q) + \frac{\mu_{cr}^2}{f'(0)}V_*\right], \tag{45}$$

the second equality following from the obvious relation $t_* = gV_*/f'(0)$ implied by (44), and the definition of g^2 in (20). Moreover, for $s_* \geq 0$ (37) implies that

$$\frac{d\Phi(s)}{ds}\bigg|_{s=s_*} = \frac{1}{2}\left[s_* - \theta(s_* - 2)\sqrt{s_*^2 - 4}\right]. \tag{46}$$

Assuming first $0 \leq s_* \leq 2$, the first of the equations (44) together with (45), (46) immediately yields

$$V_* = \mu(q)\frac{f'(0)}{\mu_{cr}^2}, \qquad s_* = 2\mu(q)\frac{1}{\mu_{cr}}, \quad \text{and} \quad t_* = \mu(q)\frac{g}{\mu_{cr}^2}. \tag{47}$$

Obviously, this solution is compatible with our assumption $0 \leq s_* \leq 2$ only as long as $0 \leq \mu(q) \leq \mu_{cr}$. On the other hand, assuming $s_* > 2$ the first of the equations (44) solved together with (46) yields the relation

$$s_* = \frac{\mu_{cr}}{f'(0)}V_* + \left(\frac{\mu_{cr}}{f'(0)}V_*\right)^{-1}, \tag{48}$$

which is consistent with our assumption. Substituting for s_* from (45) one immediately finds

$$V_* = \frac{f'(0)}{\mu(q)}, \qquad t_* = \frac{g}{\mu(q)} \quad \text{for } \mu(q) > \mu_{cr}. \tag{49}$$

In fact, V_* is nothing else but the most probable value of the potential $V(\mathbf{x})$ at a stationary point of the energy surface $\mathcal{H}(\mathbf{x})$ situated at the distance $|\mathbf{x}| = \sqrt{NR}$ from the origin. We see that V_* experiences a drastic change in its behaviour at the values of radial parameter q at which the curve $\mu(q)$ crosses the value $\mu_{cr} = \sqrt{f''(0)}$ and zero. To simplify our analysis we assume in the following that $\mu(q) > 0$ and $\mu'(q) \geq 0$, $\forall q \geq 0$, i.e. that $\mu(q)$ is a non-decreasing function of its argument. Then the equation $\mu(q) = \mu_{cr}$ has either a single solution $q = q_*$ when $\mu(0) < \mu_{cr} < \mu(q \to \infty)$, or no solution at all (this happens if $\mu_{cr} < \mu(0)$, or $\mu_{cr} > \mu(\infty)$).

The corresponding values of $\mathcal{S}(V_*, t_*, q)$ can be found by substituting (47) and (49) into (42). In particular, the expression (48) in the regime (49) implies the identity $\frac{1}{2}[s_* - \sqrt{s_*^2 - 2}] = \frac{\mu_{cr}}{\mu(q)}$, so that $\Phi(s_*) = \frac{\mu_{cr}^2}{2\mu^2} + \frac{1}{2} - \ln\frac{\mu_{cr}}{\mu(q)}$, and we obtain after some simple algebra:

$$\mathcal{S}(V_*, t_*, q) = \begin{cases} -\dfrac{1}{2} + \ln\left(\dfrac{\mu_{cr}}{\mu(q)}\right), & \mu(q) \geq \mu_{cr}, \\ -\dfrac{\mu^2(q)}{2\mu_{cr}^2}, & \mu(q) < \mu_{cr}. \end{cases} \tag{50}$$

Substituting $\mathcal{F}(q) \propto \exp[-N\mathcal{S}(V_*, t_*, q)]$ back to the integral (41), we see that the total number of stationary points is given by

$$\langle \#_s \rangle = \mathcal{N}_s \int_0^\infty e^{\frac{N}{2}\Psi(q)} \frac{dq}{q}, \tag{51}$$

where $\mathcal{N}_s$ encapsulates all the necessary normalisation factors, and

$$\Psi(q) = \begin{cases} \dfrac{q\mu^2(q)}{f'(0)} + 1 - \ln\left(\dfrac{\mu_{cr}^2}{\mu^2(q)}\right) + \ln q, & \text{for } \mu(q) \geq \mu_{cr}, \\ \dfrac{q\mu^2(q)}{f'(0)} + \dfrac{\mu^2(q)}{\mu_{cr}^2} + \ln q, & \text{for } \mu(q) < \mu_{cr}. \end{cases} \tag{52}$$

To find $\mathcal{N}_s$ it suffices to consider the special case where $\mu(q) \to \infty$ when only the minimum at the origin survives and hence $\langle \#_s \rangle \to 1$. This condition fixes with exponential in N accuracy

$$\mathcal{N}_s \sim e^{-N \ln R_{cr}}, \tag{53}$$

where the scale R_{cr} was defined in (20). Taking this factor into account, and assuming that the equation $\mu(q) = \mu_{cr}$ has a single solution $q = q_*$, we split the integration range in (51) into two natural pieces $0 < q \leq q_*$ and $q > q_*$. Then the exponential in N part of the mean number of stationary points can be found from

$$\langle \#_s \rangle = \int_0^{q_*} e^{\frac{N}{2}\Psi_<(q)} \frac{dq}{q} + \int_{q_*}^\infty e^{\frac{N}{2}\Psi_>(q)} \frac{dq}{q}, \tag{54}$$

where remembering $f'(0) = -\mu_{cr}^2 R_{cr}^2$ we defined

$$\begin{cases} \Psi_>(q) = 1 - w + \ln w, \text{ where } w = \dfrac{q\mu^2(q)}{\mu_{cr}^2 R_{cr}^2}, & \text{for } \mu(q) \geq \mu_{cr} \equiv \mu(q_*), \\ \Psi_<(q) = w\dfrac{R_{cr}^2}{q}\left(1 - \dfrac{q}{R_{cr}^2}\right) + \ln\left(\dfrac{q}{R_{cr}^2}\right), & \text{for } \mu(q) < \mu_{cr} \equiv \mu(q_*). \end{cases} \tag{55}$$

As the maximum of the function $\Psi_>(q) = 1 - w + \ln w$ is achieved at $w = 1$ and equal to zero we have $\Psi_>(q) \leq 0, \forall q > q_*$, and hence only the first integral in (54) can yield a positive complexity

$$\Sigma_s = \lim_{N\to\infty} \frac{1}{N} \ln \langle \#_s \rangle = \lim_{N\to\infty} \frac{1}{N} \ln\left(\int_0^{q_*} e^{\frac{N}{2}\Psi_<(q)} \frac{dq}{q}\right). \tag{56}$$

To extract the large-N asymptotics of the latter integral we calculate

$$\frac{d\Psi_<(q)}{dq} = \frac{dw}{dq}\left(\frac{R_{cr}^2}{q} - 1\right) + \frac{1}{q}\left(1 - \frac{\mu^2(q)}{\mu_{cr}^2}\right), \quad \mu(q) < \mu(q_*) = \mu_{cr}. \tag{57}$$

The analysis below proceeds separately for two cases: $0 < q_* < R_{cr}^2$ and $0 < R_{cr}^2 < q_*$.

1. Let us first consider $q_* < R_{cr}^2$, with the goal to demonstrate that for such a choice there is no contribution to positive complexity. We have $\frac{dw}{dq} > 0$, so that the derivative in (57) is obviously positive: $\frac{d\Psi_<(q)}{dq} > 0$, $q < q_*$. Using that $w_* = w(q_*) = \frac{q_*}{R_{cr}^2} < 1$ we see $\Psi_<(q) < \Psi_<(q_*) = 1 - w_* + \ln w_* < 0$ and conclude that in such a case there is no exponentially large contribution to the integral, hence no positive complexity.
2. Assume now $0 < R_{cr}^2 < q_*$. We shall demonstrate the existence of a point $q_0 \in [R_{cr}^2, q_*]$ where the function $\Psi_<(q)$ attains a positive maximum $\Psi_<(q_0) > 0$. This in turn will imply a positive complexity.
   To begin with, $R_{cr}^2 < q_*$ implies $\mu(R_{cr}^2) < \mu(q_*) = \mu_{cr}$ as the function $\mu(q)$ is non-decreasing. Then (57) implies $\frac{d\Psi_<(q)}{dq}|_{q=R_{cr}^2} = \frac{1}{R_{cr}^2}\left(1 - \frac{\mu^2(R_{cr}^2)}{\mu_{cr}^2}\right) > 0$. Note that from (55) follows $\Psi_<(q < R_{cr}^2) < 0$, and $\Psi_<(q = R_{cr}^2) = 0$. Therefore $\Psi_<(q) > 0$ in some right vicinity of $q = R_{cr}^2$.
   On the other hand, we have already seen above that $\Psi_<(q_*) < 0$ and also (57) implies

$$\left.\frac{d\Psi_<(q)}{dq}\right|_{q=q_*} = \left.\frac{dw}{dq}\right|_{q_*}\left(\frac{q_{cr}^2}{q_*} - 1\right) < 0. \tag{58}$$

   Then by continuity there must exists a point $q = q_1 \in (R_{cr}^2, q_*)$ such that $\Psi_<(q_1) = 0$. Remembering $\Psi_<(q = R_{cr}^2) = 0$ and positivity of $\Psi_<(q)$ to the immediate right of $q = R_{cr}^2$ we conclude on the existence of another point $q = q_0 \in (R_{cr}^2, q_1)$ where the function $\Psi_<(q)$ attains at least one positive maximum $\Psi_<(q_0) > 0$.

Combining all these facts, we see that under these conditions, (56) leads to the positive cumulative complexity of stationary points:

$$\Sigma_s = \frac{1}{2}\max_{q\in(R_{cr}^2, q_*)}\left[\frac{\mu^2(q)}{\mu_{cr}^2}\left(1 - \frac{q}{R_{cr}^2}\right) + \ln\left(\frac{q}{R_{cr}^2}\right)\right] > 0, \tag{59}$$

as long as $\mu(R_{cr}^2) < \mu_{cr}$. Taking into account definitions (20), the latter condition can be written in terms of the covariance function of the random potential as

$$\mu\left(-\frac{f'(0)}{f''(0)}\right) < \sqrt{f''(0)}. \tag{60}$$

This inequality is nothing else but precisely the condition of instability of the zero-temperature replica symmetric solution of the associated problem in statistical mechanics, see (152) in the Appendix 4.

For the simplest case of a parabolic confinement (6) with $\mu(q) = \mu < \mu_{cr}$ we can easily find the cumulative complexity as an explicit function of μ. Indeed, the point where $\Psi_<(q)$ attains its maximum is given by $q_0 = R_{cr}^2\frac{\mu_{cr}^2}{\mu^2}$, and

$$\Sigma_s = \frac{1}{2}\left[-1 - \ln\frac{\mu^2}{\mu_{cr}^2} + \frac{\mu^2}{\mu_{cr}^2}\right] \geq 0, \quad \text{for } \mu \leq \mu_{cr} \tag{61}$$

in full agreement with the result reported earlier in [51]. When approaching the critical value $\mu = \mu_{cr}$ the cumulative complexity vanishes quadratically: $\Sigma_s \approx (1 - \frac{\mu}{\mu_{cr}})^2$.

Let us show that such type of critical behaviour is generic. Introducing the short-hand notations $\tilde{q} = q/R_{cr}^2$ and $\tilde{\mu}(\tilde{q}) = \frac{\mu(q)}{\mu_{cr}}$, we rewrite $\Psi_<(q)$ as

$$\Psi_<(\tilde{q}) = (1 - \tilde{q})\tilde{\mu}^2(\tilde{q}) + \ln \tilde{q}. \tag{62}$$

The condition of criticality (60) is simply $\mu(R_{cr}^2)/\mu_{cr} = 1$, which amounts in new notations to $\tilde{\mu}(1) = 1$. Hence we introduce the parameter $\delta = 1 - \tilde{\mu}(1)$ which controls the distance to the criticality. Obviously, the interval $R_{cr}^2 \le q \le q_*$ shrinks to zero when $\delta \to 0$, so everywhere in the critical region $\tilde{q} - 1 = \epsilon \ll 1$, and we can approximate the function $\tilde{\mu}(\tilde{q})$ by first two terms in Taylor expansion: $\tilde{\mu}(\tilde{q}) \approx \tilde{\mu}(1) + \epsilon\tilde{\mu}'(1) \equiv 1 - \delta + \epsilon\tilde{\mu}'(1)$. Substituting this to (62) we obtain

$$\Psi_<(\tilde{q} = 1 + \epsilon) = (2\delta + O(\delta^2))\epsilon - \left(\frac{1}{2} + 2\tilde{\mu}'(1) + O(\delta)\right)\epsilon^2 + O(\epsilon^3). \tag{63}$$

To find the complexity, we should maximize this over ϵ. To the leading order in δ the maximum is attained at $\epsilon_0 = \delta/(\frac{1}{2} + 2\tilde{\mu}'(1))$ and the corresponding complexity is given by

$$\Sigma_s = \frac{1}{2}\Psi_<(\tilde{q} = 1 + \epsilon_0) = \frac{\delta^2}{1 + 4\tilde{\mu}'(1)}, \tag{64}$$

which indeed always vanishes quadratically at criticality.

2.3 The Density of Minima for Smooth Spherically-Symmetric Potentials, and the Associated Complexity

If one is interested in calculating the density of only minima, one has to perform the manipulations identical to those in the first half of the preceding section leading to (31) and (35), but using (5) rather than (4) as a starting point. Repeating all the steps, one arrives then to

$$\rho_m(V, \mathbf{x}, [U]) = \mathcal{N}_m \exp N\left(\frac{\mathbf{x}^2}{2f'(0)}\mu^2(\mathbf{x}^2) - \frac{V^2}{2f(0)}\right) \int dt e^{-Nt^2/2} \mathcal{D}_m(t, V, [U]) \tag{65}$$

with

$$\mathcal{D}_m(t, V, [U]) \equiv D_m(s) = \langle \det(\hat{H} + s\hat{I})\theta[\det(\hat{H} + s\hat{I})]\rangle_{GOE}, \tag{66}$$

where the variable s is given in terms of t and V by the same expression (30), and $\mathcal{N}_m$ is a relevant normalisation constant. As before we use the notation $\langle \ldots \rangle_{GOE}$ for the average with respect to the standard GOE measure $\mathcal{P}(\hat{H})\, d\hat{H} \propto \exp(-N\mathrm{Tr}\hat{H}^2/4)\, d\hat{H}$. To extract large-$N$ behaviour of the function $D_m(s)$ we employ again the Dean–Majumdar method as explained in the Appendix 3. In this way one finds that the function $D_m(s)$ with the required accuracy is given asymptotically by

$$D_m(s) \propto \begin{cases} \exp\left\{-\dfrac{N^2}{2}G_m(s) + NT_m(s)\right\}, & \text{if } s < 2, \\ \exp[N\phi(s)], & \text{if } s > 2, \end{cases} \tag{67}$$

where the calculation sketched in the Appendix 3 gives

$$G_m(s) = \frac{1}{216}\left(72s^2 - s^4 - 30s\sqrt{12+s^2} - s^3\sqrt{12+s^2}\right) - \ln\frac{(s+\sqrt{s^2+12})}{6}, \tag{68}$$

so that $G_m(2) = 0$, and $\phi(s)$ is the same as expression (37) for $s > 2$, i.e.

$$\phi(s) = \frac{s^2}{4} - \left[\frac{s\sqrt{s^2-4}}{4} + \ln\left(\frac{s-\sqrt{s^2-4}}{2}\right)\right]. \tag{69}$$

Explicit expression for the function $T_m(s)$ can be easily found from the procedure described in the Appendix 3, but is immaterial for our purposes, apart from the fact that continuity of $\mathcal{D}(s)$ at $s = 2$ and the property $G_m(2) = 0$ implies $T_m(2) = \phi(2) = 1$.

Correspondingly, the density of minima in (65) is calculated as the sum of two contributions:

$$\rho_m^{(1)} = \mathcal{N}_m \exp N\left(\frac{\mathbf{x}^2}{2f'(0)}\mu^2(\mathbf{x}^2) - \frac{V^2}{2f(0)}\right) \times \int_{-\infty}^{\infty} \exp\left(-\frac{Nt^2}{2} + N\,T_m(s) - \frac{N^2}{2}G_m(s)\right)\theta(2-s)\,dt, \tag{70}$$

$$\rho_m^{(2)} = \mathcal{N}_m \exp N\left(\frac{\mathbf{x}^2}{2f'(0)}\mu^2(\mathbf{x}^2) - \frac{V^2}{2f(0)}\right) \times \int_{-\infty}^{\infty} \exp\left(-\frac{Nt^2}{2} + N\phi(s)\right)\theta(s-2)\,dt. \tag{71}$$

To shorten our analysis in the large-N limit we recall that our final goal is rather to calculate the exponential in N contribution to the mean number of minima. As in this process we integrate the above density over the value of the potential V, and over the coordinates $\mathbf{x}$ in the sample, we can search simultaneously for the optimal values of variables t and V. Let us start with the asymptotic analysis of the second contribution, (71). The shape of the integrand in this case coincides with that in the analysis of the density of stationary points for $s > 2$, hence the optimal values are given by (see (49))

$$V_* = \frac{f'(0)}{\mu(q)}, \quad t_* = \frac{g}{\mu(q)} \quad \text{for } \mu(q) > \mu_{cr}, \tag{72}$$

where as before we introduced $q = \mathbf{x}^2$. In particular, the most probable values V_* of the potential $V(\mathbf{x})$ taken over all the stationary point, or over only minima coincide in this regime. The corresponding contribution to the density of minima is given by

$$\rho_m^{(2)} \propto \exp\frac{N}{2}\left(\frac{q}{2f'(0)}\mu^2(q) - \ln\frac{\mu_{cr}^2}{\mu^2(q)}\right). \tag{73}$$

This is again the same as contribution of the density of all stationary points in the regime $\mu(q) > \mu_{cr}$. As we know from previous analysis, in this regime there can be no exponentially many stationary points, even less so minima. Similar consideration also makes it clear that the overall normalization factor $\mathcal{N}_m$ (common to both $\rho_m^{(1)}$ and $\rho_m^{(2)}$) in the final expression should be (within exponential accuracy) the same as one used for the total density of stationary points in (53).

Now we consider the contribution coming from (70), which can be conveniently rewritten using the relation between t and s as

$$\rho_m^{(1)} \propto \exp N\left(\frac{q}{2f'(0)}\mu^2(q) - \frac{V^2}{2f(0)}\right) \times \int_{-\infty}^{2} \exp -N\left(\frac{1}{2g^2}\left[\mu_{cr}s - \mu(q) - \frac{f'(0)}{f(0)}V\right]^2 - T_m(s) + \frac{N}{2}G_m(s)\right)ds. \tag{74}$$

When $N \to \infty$, the integral over s will be obviously dominated by the value which minimizes the function $G_m(s)$. One can satisfy oneself that this minimum occurs precisely at the boundary of the integration region $s = 2$. Defining $s = 2 - z$ we expand for small z as $G_m(2-z) \approx \frac{z^3}{12} + \mathcal{O}(z^4)$, and after substituting this expansion back to (74) integrate out z. This procedure yields an irrelevant pre-exponential factor, and using $\phi(2) = 1$ the exponential in N contribution to the density takes the form

$$\rho_m^{(1)} \propto \exp \frac{N}{2}\left(\frac{q}{f'(0)}\mu^2(q) - \frac{V^2}{f(0)} + 2 - \frac{1}{g^2}\left[2\mu_{cr} - \mu(q) - \frac{f'(0)}{f(0)}V\right]^2\right). \tag{75}$$

Next we find that this expression is maximized at the value of the potential given by

$$V_* = \frac{f'(0)}{\mu_{cr}}\left[2 - \frac{\mu(q)}{\mu_{cr}}\right], \tag{76}$$

and the value of the density $\rho_m^{(1)}$ at this maximum is proportional to

$$\rho_m^{(1)} \propto \exp \frac{N}{2}\left(\frac{q}{f'(0)}\mu^2(q) + 2 - \left[2 - \frac{\mu(q)}{\mu_{cr}}\right]^2\right). \tag{77}$$

Now this expression can be used for extracting the exponential in N contribution to the total mean number $\langle \#_m \rangle$ of minima, hence the corresponding complexity. This amounts to multiplying the density (77) with the "volume" factor $q^{(N-2)/2}dq$, integrating over the radial variable R in the range up to $q = q_*$, such that $\mu(q) < \mu_{cr}$ for $q < q_*$, and finally multiplying with the overall normalisation factor $e^{-N \ln R_{cr}}$. Taking the logarithm yields the complexity of minima

$$\Sigma_m = \lim_{N\to\infty} \frac{1}{N} \ln\langle \#_m \rangle = \lim_{N\to\infty} \frac{1}{N} \ln\left(\int_0^{R_*} e^{\frac{N}{2}\Psi_m(q)} \frac{dq}{q}\right) \tag{78}$$

$$= \frac{1}{2} \max_{q\in(0,q_*)} \Psi_m(q), \tag{79}$$

where

$$\Psi_m(q) = 2 + \ln \frac{q}{R_{cr}^2} - \frac{q}{R_{cr}^2}\frac{\mu^2(q)}{\mu_{cr}^2} - \left[2 - \frac{\mu(q)}{\mu_{cr}}\right]^2. \tag{80}$$

Let us first consider the simplest case of a parabolic confinement $\mu(q) = \mu < \mu_{cr}$, see (6). The complexity of minima can be easily found as the function of μ/μ_{cr}. Again, the point q_0 at which $\Psi_m(q)$ attains its maximum is equal to the same value $q_0 = R_{cr}^2 \frac{\mu_{cr}^2}{\mu^2}$ which

delivered earlier the maximum to the function $\Psi_<(R)$, and

$$\Sigma_m\left(\frac{\mu}{\mu_{cr}}\right)=\frac{1}{2}\left[-3-\ln\frac{\mu^2}{\mu_{cr}^2}+4\frac{\mu}{\mu_{cr}}-\frac{\mu^2}{\mu_{cr}^2}\right],\quad \text{for } \mu\le\mu_{cr}. \tag{81}$$

It is evident that $\Sigma_m(\frac{\mu}{\mu_{cr}}=1)=0$, and it is easy to check that $d\Sigma_m/d\mu<0$, hence $\Sigma_m(\frac{\mu}{\mu_{cr}}=1)>0$ for any $\frac{\mu}{\mu_{cr}}<1$. Close to the critical value $\frac{\mu}{\mu_{cr}}=1$ the complexity Σ_m vanishes cubically: $\Sigma_m\approx\frac{1}{3}(1-\frac{\mu}{\mu_{cr}})^3$. This is a faster decrease in comparison with the quadratic behaviour demonstrated by the cumulative complexity from (61). In fact note that for any $\mu\le\mu_{cr}$ the inequality $\Sigma_s-\Sigma_m=(1-\frac{\mu}{\mu_{cr}})^2>0$ holds.

In a general case $\mu'(q)>0$ the point q_0 at which the function $\Psi_<(q)$ from (55) attains its maximum does not necessarily coincide with one which delivers the maximum to $\Psi_m(q)$ from (80). Indeed, the two functions satisfy the following relation:

$$\Psi_m(q)=\Psi_<(q)-2\left[1-\frac{\mu(q)}{\mu_{cr}}\right]^2, \tag{82}$$

which implies for derivatives

$$\frac{d}{dq}\Psi_m(q)=\frac{d}{dq}\Psi_<(q)+4\frac{\mu'(q)}{\mu_{cr}}\left[1-\frac{\mu(q)}{\mu_{cr}}\right]. \tag{83}$$

Hence in the regime $\mu(q)<\mu_{cr}$ interesting for us here $\frac{d}{dq}\Psi_m(q)>0$ at the point of maximum of the function $\Psi_<(q)$.

As discussed in the previous section, for $q_\star<R_{cr}^2$ the derivative $\frac{d}{dq}\Psi_<(q)$ is strictly positive, and thus the derivative $\frac{d}{dq}\Psi_m(q)$ cannot be zero. This implies that $\Psi_m(q)$ can only have a maximum if the inequality $q_*>R_{cr}^2$ holds, and the point of maximum must belong to the interval $q\in[R_{cr}^2,q_*]$. Below we shall prove that in fact the condition $q_*>R_{cr}^2$ is enough to ensure that complexity of minima is positive. Let us however note that the simple arguments used to prove the positivity of the maximum of $\Psi_s(R)$, hence the positivity of the cumulative complexity Σ_s, are not immediately applicable in the case of the complexity of minima. Indeed, although we have $\frac{d}{dq}\Psi_m(q)|_{R=R_{cr}^2}=[1-(\frac{\mu(R_{cr}^2)}{\mu_{cr}^2})^2]>0$, we still have $\Psi_m(q=R_{cr}^2)<0$ in view of the relation (82) and $\Psi_<(q=R_{cr}^2)=0$. But we also have at the right end of the interval $\Psi_m(q=q_*)<0$ and $\frac{d}{dq}\Psi_m|_{q_*}<0$, demonstrating impossibility to infer positivity of the maximum of $\Psi_m(q)$ inside (R_{cr}^2,q_*) along these lines.

To circumvent this difficulty, we introduce an auxiliary function $\Psi_a(\tilde q,\tilde\mu)$ of *two* real variables $\tilde q$ and $\tilde\mu$ according to

$$\Psi_a(\tilde q,\tilde\mu)=2+\ln\tilde q-\tilde q\tilde\mu^2-[2-\tilde\mu]^2. \tag{84}$$

Note that $\Psi_m(q)$ in (80) is obtained from the above function by replacing its arguments as $\tilde q\to q/R_{cr}^2$ and $\tilde\mu\to\frac{\mu(q)}{\mu_{cr}}$.

Considering values of $\Psi_a(\tilde q,\tilde\mu)$ we easily find that the function can take positive values only in a wedge-like region of the $(\tilde q,\tilde\mu)$ plane restricted by two boundary curves $\tilde\mu_-(q)$ and $\tilde\mu_+(q)$, where

$$\tilde\mu_\pm(\tilde q)=\frac{2\pm\sqrt{D(\tilde q)}}{1+\tilde q},\quad D(\tilde q)=4+(1+\tilde q)(\ln(\tilde q)-2), \tag{85}$$

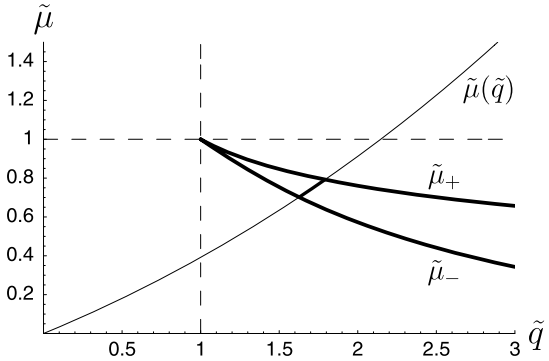

Fig. 1 A representative function $\tilde{\mu}(\tilde{q})$ which is monotonically increasing and satisfies the condition $\tilde{q}_* > 1$. These conditions are sufficient to ensure that the corresponding $\Psi_m(q)$ has a positive maximum leading to a positive complexity of minima

and one has to require $D(\tilde{q}) \geq 0$ for the existence of these curves. Since $D(\tilde{q}=1)=0$ and $\frac{dD(\tilde{q})}{d\tilde{q}} = -1 + \ln q + \frac{1}{q} \geq 0$, $\forall q > 0$, we conclude that $D(\tilde{q} \geq 1) \geq 0$. This implies the region in between those two curves exists for $\tilde{q} > 1$, and for $\tilde{q} \to 1$ the two boundary curves approach each other and meet at the point $\tilde{\mu}_\pm(\tilde{q}=1)=1$ (see Fig. 1).

Now, the problem of finding the complexity of minima, (78), for any monotonically increasing confinement function $\mu(q)$ is equivalent to searching a maximum of the function $\Psi_a(\tilde{q},\tilde{\mu})$ along the corresponding curve $\tilde{\mu}(\tilde{q}) = \mu(q)/\mu_{cr}$ in the plane (see Fig. 1). The point q_* appearing in our analysis is nothing else but the point of intersection of that curve with the horizontal line $\tilde{\mu}=1$, and the condition $q_* > R_{cr}^2$ just means that such an intersection happens at some value $\tilde{q}_* > 1$. But every such curve $\tilde{\mu}(\tilde{q})$ necessarily intersects the region between the boundary curves $\tilde{\mu}_\pm(\tilde{q})$, and therefore necessarily has a portion along which the values of $\Psi_a(\tilde{q},\tilde{\mu}) > 0$. This proves that the complexity of minima is always positive as long as the condition $q_* > R_{cr}^2$ holds. This is equivalent to $\mu(R_{cr}^2) < \mu_{cr}$ which is precisely the replica symmetry breaking condition (60).

Let us finally demonstrate that for a generic smooth confining potential the complexity of minima vanishes always cubically at criticality, the type of critical behaviour we have already found in the simplest case of parabolic confinement. For this we follow the same steps as in the end of the preceding section. Using the parameter $\delta = 1 - \tilde{\mu}(1) \ll 1$ which controls the distance to the criticality, we approximate in the critical region $\tilde{q} = 1+\epsilon$, $\epsilon \ll 1$, and use for the function $\tilde{\mu}(\tilde{q})$ its Taylor expansion. Truncating as before at first order: $\tilde{\mu}(\tilde{q}) \approx 1 - \delta + \epsilon\tilde{\mu}'(1)$ and substituting this for $\tilde{\mu}$ in (84), we obtain

$$\Psi_m(\tilde{q}=1+\epsilon) = -2\delta^2 + [2\delta + 4\tilde{\mu}'(1)\delta - \delta^2)]\epsilon - \left[\frac{1}{2} + 2\tilde{\mu}'(1) + 2(\tilde{\mu}'(1))^2 + O(\delta)\right]\epsilon^2 + O(\epsilon^3). \tag{86}$$

To find the corresponding complexity, we should maximize this over ϵ. To the leading order in δ the maximum is attained at $\epsilon_0 = 2\delta/(1+2\tilde{\mu}'(1))$. Substituting this value back to (86) we immediately find that all terms of the order of δ^2 cancel, and thus at criticality the complexity of minima behaves at least as $\Sigma_s = C\delta^3$. To prove that the coefficient $C \neq 0$ and to find its actual value in terms of $\tilde{\mu}'(1)$, $\tilde{\mu}''(1)$ requires going to the terms of order ϵ^2 in the Taylor expansion for $\tilde{\mu}(\tilde{q})$, and to the order ϵ^3 in (86). This is a straightforward but boring exercise, and the resulting expression is not very elegant apart from the simplest case of the parabolic confinement when $\tilde{\mu}'(1) = \tilde{\mu}''(1) = 0$, and $C = 1/3$ as we have already found before.

2.4 On Anomalous Critical Behavior of Complexities for Hard-Wall Confining Potentials

A fairly universal type of the critical behavior of both complexities Σ_s and Σ_m revealed in the preceding sections should be compared with the results reported recently in our short communication [55], and independently in [54]. In both papers the landscape was confined in a finite box, chosen in [55] to be spherical of extent $|\mathbf{x}| \leq L = R\sqrt{N},\ 0 < R < \infty$. On top of such a hard-wall confinement a harmonic confining potential was superimposed to allow a comparison with the results of statistical mechanics of the same problem studied by the replica trick in [15]. For the sake of clarity, we will consider below only pure hard-wall confinement, when the complexities for a given value of the radius R were found to be given by [54, 55]:

$$\Sigma_s(R) = \ln\left(\frac{R}{R_{cr}}\right) \quad \text{for } R \geq R_{cr}, \quad \text{and} \quad \Sigma_s(R) \leq 0 \text{ otherwise}, \tag{87}$$

and

$$\Sigma_m(R) = -1 + \ln\left(\frac{R}{R_{cr}}\right) \quad \text{for } R \geq R_m = eR_{cr}, \quad \text{and} \quad \Sigma_m(R) \leq 0 \text{ otherwise}, \tag{88}$$

with R_{cr} given by (20). Taking into account that the domain of zero-temperature replica symmetry breaking in this particular model is just given by $R > R_{cr}$ [15], we see that apparently the type of behavior exemplified by (87), (88) is very different from what we have discussed earlier in this paper. Namely, although in this case the cumulative complexity Σ_s is also positive everywhere in the phase with broken replica symmetry for $R > R_{cr}$, the complexity of minima Σ_s becomes positive only starting from a larger confining radius $R_m = eR_{cr} > R_{cr}$. In other words, for the interval $R_{cr} < R < R_m$ the broken ergodicity is not at all accompanied by the exponentially many minima in the energy landscape. Another peculiarity is that the complexity Σ_s vanishes linearly rather than quadratically with the distance $\delta_R = R/R_c - 1$ close to the ergodicity threshold.

In the framework of the present approach the function $\mu(q)$ describing the hard-wall confinement with a radius R is formally $\mu(q) = 0$ for $q < R^2$ and $\mu(q) = \infty$ for $q \geq R^2$. Such a form is apparently highly singular, and does not immediately fit into the analysis of the preceding sections. To circumvent this difficulty and to understand better the origin of the hard-wall behavior (87), (88) within a more general framework we consider a family of non-smooth confining potentials of the form:

$$\mu(q) = \begin{cases} 0, & \text{for } q < R^2, \\ \mu\sqrt{h\left(\dfrac{q}{R^2}\right)}, & \text{for } q \geq R^2, \end{cases} \tag{89}$$

where $\mu > 0$ is a control parameter, and $h(x)$ is an increasing, non-negative smooth concave function $h(x) > 0, h'(x) \geq 0, h''(x) \geq 0$ for $\forall x > 1$. We also assume $h(1) = 0$ to ensure the continuity of $\mu(q)$, and thus existence of a single solution $q = q_*$ of the equation $\mu(q) = \mu_{cr}$ which as we know plays an important role in our analysis. We will be in particular interested in understanding the behavior of complexities $\Sigma_s(R)$ and $\Sigma_m(R)$ in the limit $\mu \gg 1$, where the confinement described by (89) should approach a hard-wall form. We also assume the confinement radius to satisfy $R > R_{cr}$, to allow for a positive complexity.

The cumulative complexity of stationary points for this class of potentials is given according to (59) by $\Sigma_s^{(\mu)}(R) = \frac{1}{2}\max_{\tilde{q}\in(1,q_*)} \Psi_s^{(\tilde{\mu})}(\tilde{R},\tilde{q})$ where

$$\Psi_s^{(\tilde{\mu})}(\tilde{R},\tilde{q}) = \begin{cases} \ln\tilde{q}, & \text{for } 0<\tilde{q}<\tilde{R}^2, \\ \ln\tilde{q} + \tilde{\mu}^2 h(\tilde{q}/\tilde{R}^2)(1-\tilde{q}), & \text{for } \tilde{R}^2 \le \tilde{q} \le \tilde{q}_*, \end{cases} \tag{90}$$

and we used as usual the scaled parameter $\tilde{\mu} = \mu/\mu_{cr}$ and the scaled confinement radius $\tilde{R} = R/R_{cr} > 1$ as well as $\tilde{q} = q/R_{cr}^2$, with the value of the parameter $\tilde{q}_*$ being fixed by the condition $\tilde{\mu}^2 h(\tilde{q}_*/\tilde{R}^2) = 1$. Differentiating the expression (90) over $\tilde{q}$ gives:

$$\left.\frac{d\Psi_s}{d\tilde{q}}\right|_{\tilde{q}\to\tilde{R}^2+} = \frac{1}{\tilde{R}^2} - \tilde{\mu}^2[h'(1)(\tilde{R}^2-1)], \tag{91}$$

which is clearly negative for large enough $\tilde{\mu}$, except for the special case $h(1) = h'(1) = 0$. On the other hand, $\frac{d\Psi_s}{d\tilde{q}}|_{0<\tilde{q}<\tilde{R}^2} = \frac{1}{\tilde{R}^2} > 0$. We conclude that the function $\Psi_s^{(\tilde{\mu})}(\tilde{R},\tilde{q})$ for large enough μ has its local maximum precisely at $\tilde{q} = \tilde{R}^2$. Actually, this is the global maximum, as

$$\left.\frac{d^2\Psi_s}{d\tilde{q}^2}\right|_{\tilde{q}>\tilde{R}^2} = -\frac{1}{\tilde{R}^4}\{1 + \tilde{\mu}^2[2\tilde{R}^2 h'(\tilde{q}/\tilde{R}^2) + h''(\tilde{q}/\tilde{R}^2)(\tilde{q}-1)]\} < 0, \tag{92}$$

so that $\frac{d\Psi_s}{d\tilde{q}} < \frac{d\Psi_s}{d\tilde{q}}|_{\tilde{q}\to\tilde{R}^2+} < 0$ for all $\tilde{R}^2 < \tilde{q} < \tilde{q}_*$. We immediately see that the cumulative complexity is then given simply by $\frac{1}{2}\ln\tilde{R}^2$.

Let us now shortly discuss the complexity of minima along the same lines. The analogue of the expression (90) follows from (80), and is given by

$$\Psi_m^{(\tilde{\mu})}(\tilde{R},\tilde{q}) = \begin{cases} \ln\tilde{q} - 2, & \text{for } 0<\tilde{q}<\tilde{R}^2, \\ \ln\tilde{q} - 2 + 4\tilde{\mu}\sqrt{h(\tilde{q}/\tilde{R}^2)} - \tilde{\mu}^2(1+\tilde{q})h(\tilde{q}/\tilde{R}^2), & \text{for } \tilde{R}^2 \le \tilde{q} \le \tilde{q}_*. \end{cases} \tag{93}$$

Differentiating the above expression, we again easily see that $\frac{d\Psi_m}{d\tilde{q}}|_{\tilde{q}\to\tilde{R}^2+} < 0$ for $\mu \gg 1$, except for the case $h(1) = h'(1) = 0$. Since $\frac{d\Psi_m}{d\tilde{q}}|_{\tilde{q}\to\tilde{R}^2-}$ is always positive, the function $\Psi_m(\tilde{R},\tilde{q})$ has its (global) maximum at $\tilde{q} = \tilde{R}^2$ and the complexity is indeed given by the expression (88).

To investigate the behavior of both complexities in the remaining exceptional case $h(1) = h'(1) = 0$, we consider a particular example $h(x) = (x-1)^2$. This implies $\tilde{q}_* = \tilde{R}^2(1+\frac{1}{\tilde{\mu}})$ and for the case of cumulative complexity we are seeking to maximize

$$\Psi_s(\tilde{q}) = \ln\tilde{q} + \tilde{\mu}^2(\tilde{q}/\tilde{R}^2 - 1)^2(1-\tilde{q}) \tag{94}$$

over $\tilde{q}$ in the interval $\tilde{R}^2 \le \tilde{q} \le \tilde{q}_*$. For large $\tilde{\mu}$ it is actually more convenient to introduce a new variable $y \in [0,1]$ via $\tilde{q} = \tilde{R}^2(1 + y/\tilde{\mu})$, and approximate $\Psi_s(\tilde{q})$ with

$$\Psi_s(y) = \ln\tilde{R}^2 + y^2(1-\tilde{R}^2) + O(1/\tilde{\mu}). \tag{95}$$

In view of the inequality $\tilde{R}^2 > 1$, the above expression attains its maximum at $y = 0$, and the cumulative complexity $\Sigma_s(R)$ is again given by the same expression (87) as in all other cases of this family in the hard-wall limit $\mu \gg 1$.

Curiously enough, the complexity of minima $\Sigma_m(R)$ in the case $h(1) = h'(1) = 0$ shows a behavior different from (88). Taking as before $h(x) = (x-1)^2$, we need this time to maximize

$$\Psi_m(\tilde{q}) = \ln \tilde{q} - 2 + 4\tilde{\mu}(\tilde{q}/\tilde{R}^2 - 1) - \tilde{\mu}^2(1+\tilde{q})(\tilde{q}/\tilde{R}^2 - 1)^2 \tag{96}$$

over $\tilde{q}$ in the interval $\tilde{R}^2 \le \tilde{q} \le \tilde{q}_* = \tilde{R}^2(1+\frac{1}{\tilde{\mu}})$. Introducing again the variable $y \in [0,1]$ via $\tilde{q} = \tilde{R}^2(1+y/\tilde{\mu})$, and assuming $\mu \gg 1$ we replace (96) with

$$\Psi_m(y) \approx \ln \tilde{R}^2 - 2 + 4y + O(1/\tilde{\mu}) - y^2(1+\tilde{R}^2), \tag{97}$$

which has its maximum at $y_0 = 2/(1+\tilde{R}^2) < 1$. The corresponding complexity is given by

$$\Sigma_m = \ln \tilde{R}^2 - 2 + \frac{4}{1+\tilde{R}^2}, \tag{98}$$

which is indeed different from (88). In particular, the last term in (98) ensures that the complexity of minima is positive everywhere in the phase with broken ergodicity $\tilde{R} > 1$, and vanishes linearly when approaching the critical value $\tilde{R} > 1$.

Thus, we have demonstrated that for every curve in the family (89) the cumulative complexity in the limit $\mu \to \infty$ is indeed given by $\Sigma_s(R) = \frac{1}{2}\ln \tilde{R}^2 \equiv \ln R/R_{cr}$, in full agreement with the hard-wall confinement formula (87). As to the complexity of minima, in the limit $\mu \to \infty$ it is generically given by (88), although the result may change if the hard-wall profile vanishes smooth enough when approaching the point of non-analyticity, $q = R^2$.

3 Conclusions and Open Questions

Let us briefly summarized our findings. We have demonstrated that for a generic, smooth concave confining potentials with a continuous positive derivative the complexity extracted from the mean number of totality of stationary points in the energy landscape is positive simultaneously with the complexity corresponding to the mean number of minima. The domain of parameters where those complexities are positive is precisely one where the zero-temperature limit of statistical mechanics in such a landscape requires for its description the concept of broken replica symmetry/broken ergodicity, (152).

On the other hand, for a non-analytic (hard-wall) confinement the boundary of non-ergodic behaviour coincides in general with the domain of positive total complexity of saddle points but not necessarily the minima. Moreover, the above analysis clearly demonstrates that all peculiar features specific to the hard-wall confinement are due to the discontinuity of the derivative of the confining potential.

In the case of a smooth confinement the complexity of minima vanishes cubically when approaching the critical confinement (60), whereas the cumulative complexity vanishes quadratically with the distance to criticality. In the Appendix 5 we follow the method by Bray and Dean [54] and investigate the (annealed) complexities of stationary points at a fixed value of the number of the negative eigenvalues of the Hessian, the so-called *index* $\mathcal{I}$ of a critical point. Restricting the consideration to the simplest case of parabolic confinement (6) we reveal that the only stationary points with non-vanishing complexity arising precisely at the critical confinement $\mu = \mu_{cr}$ are those for which the number of negative directions stays of order of unity in the large-N limit: $\alpha = \lim_{N\to\infty} \frac{\mathcal{I}}{N} = 0$. As we move inside the glassy phase away from the critical value $\mu = \mu_{cr}$, stationary points with an increasing range of indices start to have a positive complexity. This is precisely the mechanism behind

the change from cubic vanishing typical for the complexity of minima to the quadratic vanishing found for the total complexity. A more detailed characterisation of stationary points appearing precisely at the critical value $\mu = \mu_{cr}$, in particular the question of how many of them are minima, or saddle points with a fixed number (one, two, etc.) of negative eigenvalues is a very interesting question deserving further, more elaborate, investigation.

In our opinion, the results of present research suggest a few more immediate questions which are worth understanding. For example, the problem of investigating the magnitude of sample-to-sample fluctuations of the number of stationary points is clearly very interesting, though technically challenging. Another natural extension would be to consider our model at finite temperatures. This project would require defining properly an analogue of the *free* energy (TAP-like, [4]) landscape for the present model, and recovering the whole transition line to the phase with broken ergodicity from counting the corresponding stationary points. Some steps in this direction were taken in [55], but the present level of understanding remains far from being satisfactory.

Finally, let us mention that the model considered in the present paper is very intimately related [15] to the much-studied spherical model of spin glasses, which essentially corresponds to a particular choice of the confining potential $V_{con}(\mathbf{x}) = \delta(\mathbf{x}^2 - N)$ which forces the vector $\mathbf{x}$ to span the sphere of radius $\sqrt{N}$. It is natural to expect that this rather singular limit can be approached in the present method by e.g. considering the confinement of the form $U(z) = \mu(z-1)^2$ and allowing for $\mu \to \infty$. This expectation is supported by recent rigorous results [56] demonstrating that the dynamical equations known for the spherical spin glass model are faithfully reproduced in this limiting procedure. Let us however note that the above choice goes beyond the immediate scope of the present paper as we decided to restrict our attention by considering only monotonically increasing confining potentials. Actually, investigating other classes is not at all a problem, and is certainly worth doing in view of the mentioned correspondence with the spherical model, where quite a few results on complexities of extrema of free energy landscapes were already reported in the literature, see for example a good review [57] and references therein. We relegate this issue, as well as a few other possible generalisations to subsequent publications.

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Appendix 1: On a Geometric Origin of the Identity (10)

Let us start with recalling the well-known Poincare–Hopf index theorem. For any vector field $\mathbf{F}$ over any compact manifold M with no zeroes on the boundary ∂M holds the identity $Ind(\mathbf{F}) + Ind(\partial\_\mathbf{F}) = \chi(M)$, where $\chi(M)$ is a topological invariant, the Euler characteristic of the manifold. The index Ind of a vector field $\mathbf{F}$ is the sum of indices of all the singular points. i.e. sum of indices $\operatorname{sign}\det(\partial_i \mathbf{F}_j)|_{\mathbf{x}=\mathbf{x}_k}$ corresponding to all N_s isolated zeroes $\mathbf{x}_k$ of the field: $\mathbf{F}(\mathbf{x}_k) = 0$. The field $\partial\_\mathbf{F}$ is a vector field defined in terms of the values of $\mathbf{F}$ on a subset of the boundary ∂M, such that the original field $\mathbf{F}$ points inward on this subset of the boundary. Precise definition of the boundary component of the formula is immaterial for our purposes.

Consider a random surface $H(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \hat{A}\mathbf{x} + V(\mathbf{x})$, with $\mathbf{x} \in \mathbb{R}^N$, $V(\mathbf{x})$ being a mean-zero random Gaussian function with the covariance (3), and the first term being a quadratic

form involving an arbitrary non-singular real symmetric matrix $\hat{A}$, with $\det\hat{A} \neq 0$. Take any compact manifold M in $\mathbb{R}^N$ such that (i) all the stationary points $\mathbf{x}_k$ of $H(x)$ are with probability one belong to the interior of that manifold and (ii) along the boundary ∂M of the manifold the influence of random potential is already negligible (the simplest choice of M would be a ball $|\mathbf{x}| \leq L$ of a very large radius $L \to \infty$). Then topological properties of the gradient field $\mathbf{F} = \partial H(\mathbf{x})/\partial\mathbf{x}$ along the boundary ∂M are determined by the gradient of the first term, i.e. by the vector field $\mathbf{F}_A = \hat{A}\mathbf{x}$, so that $Ind(\partial_- \mathbf{F}) = Ind(\partial_- \mathbf{F}_A)$. Then the Poincare–Hopf theorem implies $Ind(\partial H(\mathbf{x})/\partial\mathbf{x}) + Ind(\partial_- \mathbf{F}_A) = \chi(M)$. On the other hand, the field $\mathbf{F}_A$ has the only zero at the origin, so that $Ind(\mathbf{F}_A) \equiv \operatorname{sign}\det\hat{A}$, and the Poincare–Hopf theorem applied to $\mathbf{F}_A$ requires $\operatorname{sign}\det\hat{A} + Ind(\partial_- \mathbf{F}_A) = \chi(M)$. Comparing these two relations we see that

$$Ind(\partial H(\mathbf{x})/\partial\mathbf{x}) \equiv \sum_{k=1}^{N_s} \operatorname{sign}\det\left(\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{x}}\right)\Bigg|_{\mathbf{x}=\mathbf{x}_k} = \operatorname{sign}\det\hat{A} \tag{99}$$

for every realization of the random surface $H(\mathbf{x})$.

On the other hand, the Dirac's δ-functional measure satisfies the fundamental identity

$$\delta\left(\frac{\partial H}{\partial\mathbf{x}}\right)\left|\det\left(\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{x}}\right)\right| = \sum_{k=1}^{N_s} \delta(\mathbf{x} - \mathbf{x}_k), \tag{100}$$

which in fact underlies the Kac–Rice formula (4). As above, the summation over $\mathbf{x}_k$ goes over all isolated zeros of the gradient $\partial H/\partial\mathbf{x}$, and $\delta(\frac{\partial H}{\partial\mathbf{x}}) \equiv \prod_{i=1}^{N} \delta(\frac{\partial H}{\partial x_i})$. This fact allows us to convert the sum in the left hand-side of (99) to an integral and rewrite (99) as the identity

$$\int_{\mathbb{R}^N} \det\left(\frac{\partial^2 H}{\partial\mathbf{x}\partial\mathbf{x}}\right)\delta\left(\frac{\partial H}{\partial\mathbf{x}}\right) d\mathbf{x} = \operatorname{sign}\det\hat{A} \tag{101}$$

valid for any realization of the random potential $V(\mathbf{x})$. We therefore can average (101) over the realizations, and take into account (i) independence of the first and second derivatives for Gaussian-distributed random functions, and (ii) stationarity of the random potential, (3). In this way we come to the relation

$$\operatorname{sign}\det\hat{A} = \left\langle\det\left(\hat{A} + \frac{\partial^2 V}{\partial\mathbf{x}\partial\mathbf{x}}\right)\right\rangle \int_{\mathbb{R}^N} \prod_{i=1}^{N} \left\langle\delta\left(\frac{\partial H}{\partial x_i}\right)\right\rangle d\mathbf{x}. \tag{102}$$

Using the standard Fourier integrals representation for the δ-functional factors, and performing the averaging over the Gaussian gradients with the covariances (cf. (21)) $\langle\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_j}\rangle = -\frac{1}{N}f'(0)\delta_{ij}$, we see that

$$\left\langle\prod_{i=1}^{N}\delta\left(\frac{\partial H}{\partial x_i}\right)\right\rangle = \int e^{-i\mathbf{u}^T\hat{A}\mathbf{x} - \frac{1}{2N}\mathbf{u}^T\mathbf{u}|f'(0)|}\frac{d\mathbf{u}}{(2\pi)^N} \tag{103}$$

$$= \frac{1}{(2\pi|f'(0)|/N)^{N/2}} e^{-\frac{N}{2|f'(0)|}\mathbf{x}^T\hat{A}^T\hat{A}\mathbf{x}}. \tag{104}$$

Substituting the last expression to (102) and performing the Gaussian integral over $\mathbb{R}^N$ yields the factor $[\det \hat{A}^T \hat{A}]^{-1/2} = 1/|\det \hat{A}|$. Then (102) is reduced to the relation

$$\det \hat{A} = \left\langle \det\left(\hat{A} + \frac{\partial^2 V}{\partial \mathbf{x} \partial \mathbf{x}} \right) \right\rangle \tag{105}$$

equivalent to the identity (10), which is thus verified for non-singular real symmetric matrices $\hat{A}$. By analytic continuation it is extended to singular case as well.

Appendix 2: Distribution of the Diagonal Element of the GOE Resolvent

Our goal is to calculate the probability distribution $\mathcal{P}(G)$ of the diagonal element of the resolvent (34) for GOE matrices $\hat{H}$, in the large-N limit, for a given real value of s.

Following the standard route we first evaluate the characteristic function

$$\chi(p) = \langle \exp\{ipG_H(\mathbf{x})\} \rangle_{GOE}. \tag{106}$$

Our first observation is that the result can depend on $\mathbf{x}$ only via the modulus $|\mathbf{x}|$ due to the rotational invariance of the GOE probability density. This implies that we can choose $\mathbf{x} = |\mathbf{x}|\mathbf{e}$, with $\mathbf{e} = (1, 0, \ldots, 0)$, so that

$$G_H(\mathbf{x}) = \mathbf{x}^2 \mathbf{e}^T \frac{1}{\hat{H} + s\hat{I}} \mathbf{e} = \mathbf{x}^2 \sum_{n=1}^{N} \frac{(\mathbf{e}, \mathbf{e}_n)^2}{s + \lambda_n}, \tag{107}$$

where $\lambda_n, \mathbf{e}_n$ stand for the eigenvalues and the corresponding eigenvectors of the GOE matrix $\hat{H}$, and $(\mathbf{e}, \mathbf{e}_n)$ is the scalar product. Remembering that the eigenvectors $\mathbf{e}_n$ and the eigenvalues λ_n of the GOE matrices are statistically independent, we perform the averaging over the eigenvectors first. To this end we recall that N GOE eigenvectors are (i) mutually orthogonal and (ii) uniformly distributed over the unit sphere $(\mathbf{e}_n, \mathbf{e}_n) = 1$. As is well-known, these conditions imply that in the large-N limit the projections $v_n = (\mathbf{e}, \mathbf{e}_n)$ behave like independent Gaussian variables with zero mean and variance $\langle v_n^2 \rangle_{GOE} = 1/N$. Denoting $\langle \ldots \rangle_v$ the averaging over these variables, we can write for the characteristic function

$$\begin{aligned} \langle \exp\{ipG_H(\mathbf{x})\} \rangle_v &= \prod_{n=1}^{N} \int_{-\infty}^{\infty} \frac{dv_n}{\sqrt{2\pi/N}} \exp\left\{ -\frac{N}{2} v_n^2 + ip\mathbf{x}^2 \frac{v_n^2}{s + \lambda_n} \right\} \\ &= \prod_{n=1}^{N} \left[\frac{N}{N - 2ip\mathbf{x}^2 \frac{1}{s+\lambda_n}} \right]^{1/2} = \prod_{n=1}^{N} \frac{[s + \lambda_n]^{1/2}}{[s - 2i\frac{p}{N}\mathbf{x}^2 + \lambda_n]^{1/2}}. \end{aligned} \tag{108}$$

Remembering the remaining averaging over the JPD of GOE eigenvalues λ_n, we see that the characteristic function (106) in the large-N limit can be written as the expectation value of the ratio of square roots of the characteristic polynomials

$$\chi(p)|_{N\to\infty} = \lim_{N\to\infty} \left\langle \frac{[\det(s + \hat{H})]^{1/2}}{[\det(s - 2i\frac{p}{N}\mathbf{x}^2 + \hat{H})]^{1/2}} \right\rangle_{GOE}. \tag{109}$$

Precisely that expectation value was already calculated earlier in a different context, see (35) and (48) of the paper [60], with the result

$$\chi(p)|_{N\to\infty} = \exp\left\{ \mathbf{x}^2 \left[\frac{i}{2} sp - |p| \pi \nu_{sc}(s) \right] \right\}, \quad \nu_{sc}(s) = \frac{1}{2\pi} \sqrt{4 - s^2}, \quad |s| < 2. \tag{110}$$

The distribution $\mathcal{P}(G)$ immediately follows from this expression after the Fourier-transform. Defining $\tilde{G} = G_H(\mathbf{x})/\mathbf{x}^2$ we see that the quantity is Cauchy-distributed:

$$\mathcal{P}(\tilde{G}) = \begin{cases} \dfrac{\nu_{sc}(s)}{\pi^2 \nu_{sc}^2(s) + (\tilde{G} - \frac{s}{2})^2}, & |s| < 2, \\ \delta\left(\tilde{G} - \dfrac{s}{2}\right), & |s| > 2. \end{cases} \tag{111}$$

Appendix 3: An Overview of the Dean–Majumdar Functional Integral Approach

Our aim in this section is the calculation of large-N asymptotics of the required GOE averages. The most economic way of arriving to the desirable expressions known to us relies on a heuristic method introduced by Dean and Majumdar [59]. We however have every reason to believe that one can arrive to the same results by employing a rigorous (and, necessarily, tedious) mathematical procedures described for a very closely related problem in a paper by A. Boutet de Monvel, L. Pastur and M. Scherbina [58].

The quantities of interest for us are the GOE averages featuring in (35) and (66), i.e.:

$$\begin{aligned} \mathcal{D}_s(s) &= \langle |\det(\hat{H} + s\hat{I})| \rangle_{\mathrm{GOE}} \\ &\propto \left[\prod_{i=1}^{N} \int_{-\infty}^{\infty} d\lambda_i \right] \prod_{i=1}^{N} |\lambda_i + s| \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j| e^{-\frac{N}{4} \sum_{i=1}^{N} \lambda_i^2}, \end{aligned} \tag{112}$$

$$\begin{aligned} \mathcal{D}_m(s) &= \langle \theta(\hat{H} + s\hat{I}) \det(\hat{H} + s\hat{I}) \rangle_{\mathrm{GOE}} \\ &\propto \left[\prod_{i=1}^{N} \int_{-s}^{\infty} d\lambda_i \right] \prod_{i=1}^{N} (\lambda_i + s) \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j| e^{-\frac{N}{4} \sum_{i=1}^{N} \lambda_i^2}, \end{aligned} \tag{113}$$

where θ-factor is equal to unity when its argument is a positive definite matrix and is zero otherwise, and λ_i stands for N real eigenvalues of the matrix $\hat{H}$. We have used that in each case the functions to be averaged depend only on the matrix eigenvalues. The average is taken over an ensemble of matrices with the probability density $\mathcal{P}(\hat{H}) d\hat{H} \propto \exp(-N \mathrm{Tr} \hat{H}^2/4)$ invariant with respect to orthogonal transformations $\hat{H} \to O \hat{H} \hat{O}^T$, and this allows us to perform in a standard way [52] the integration over $N(N-1)/2$ angular variables (eigenvectors of $\hat{H}$). The procedure yields just an overall normalization factor, and the remaining expression is given by (112). At this point it is worth mentioning, that the GOE averages of the type as above can be viewed as performed over a Gibbs measure describing an interacting gas of eigenvalues, and there are methods to developed a rigorous mean-field description of such a model in the large-N limit [58].

In a similar in spirit, but much less formal way Dean and Majumdar suggested to calculate integrals of this type by replacing the multiple integration over the eigenvalues λ_i with a functional integration over the *density* of eigenvalues, defined as

$$\rho_N(\lambda) := \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i). \tag{114}$$

Exploiting this density, we have the formal identities

$$\prod_{i=1}^{N}\phi(\lambda_i) = \exp N \int \ln\phi(\lambda)\rho_N(\lambda)\,d\lambda, \tag{115}$$

$$\prod_{i<j}^{N}\psi(\lambda_i,\lambda_j) = \exp\frac{N^2}{2}\int \ln\psi(\lambda,\lambda')\rho_N(\lambda)\rho_N(\lambda')\,d\lambda\,d\lambda', \tag{116}$$

for a suitable choice of functions $\phi(\lambda)$ and $\psi(\lambda,\lambda') = \psi(\lambda',\lambda)$.

In this way the integrands of the multivariable integrals (112) can be viewed as functionals $f[\rho]$ of the density $\rho_N(\lambda)$, and introducing the (functional) Dirac δ-function, one can formally write for such type of integrals

$$\left[\prod_{i=1}^{N}\int_a^b d\lambda_i\right] f[\rho_N] = \int \mathcal{D}[\rho] f[\rho]\left[\prod_{i=1}^{N}\int_a^b d\lambda_i\right]\delta[\rho-\rho_N], \tag{117}$$

where $f[\rho]$ is the functional f expressed in terms of the density ρ. The integrals over the eigenvalues on the right hand side of this expression can be conveniently evaluated after using the standard formal Fourier representation for the functional δ-function:

$$\delta[\rho-\rho_N] = \int \mathcal{D}[\omega]\exp\left(iN\int_{-\infty}^{\infty}\omega(t)[\rho(t)-\rho_N(t)]dt\right), \tag{118}$$

$$= \int \mathcal{D}[\omega]\exp\left(iN\int_{\infty}^{\infty}\omega(t)\rho(t)dt - i\sum_{i=1}^{N}\omega(\lambda_i)\right), \tag{119}$$

with a suitably normalized measure $\mathcal{D}[\omega]$. The integration over the eigenvalues is now trivially performed, resulting in

$$\left[\prod_{i=1}^{N}\int_a^b d\lambda_i\right]\delta[\rho-\rho_N]$$

$$\propto \int \mathcal{D}[\omega']\exp\left(iN\int_{-\infty}^{\infty}\omega'(t)\rho(t)dt + N\ln\left(\int_a^b e^{-i\omega'(t)}dt\right)\right). \tag{120}$$

For $N \gg 1$ the main contribution to the above integral should come from the stationary point of the exponent with respect to variations in the field ω' satisfying the equation

$$\rho(\lambda) = \begin{cases} \dfrac{\exp(-i\omega'(\lambda))}{\int_a^b \exp(-i\omega'(\lambda'))d\lambda'}, & \text{if } a\le\lambda\le b,\\ 0, & \text{otherwise.}\end{cases} \tag{121}$$

It is clear from the above that $\int_a^b \rho(\lambda)d\lambda = 1$, and that $-i\omega'(\lambda) = \ln\rho(\lambda) + const$ for $a \le \lambda \le b$. This yields, up to an overall constant factor, the relation

$$\left[\prod_{i=1}^{N}\int_a^b d\lambda_i\right]\delta[\rho-\rho_N] \propto \exp\left(-N\int_a^b \rho(t)\ln(\rho(t))dt\right). \tag{122}$$

Note, that looking at our system as a kind of "eigenvalue gas", this factor is simply a standard entropic contribution associated with the density of particles.

Let us apply this method for our problem, i.e. to the calculation of the required GOE averages. Denoting the range of integration in each case by $\mathcal{R}_s$ and $\mathcal{R}_m$, we have for $j \in \{s, m\}$

$$\mathcal{D}_j(s) = \int \mathcal{D}[\rho] \exp\left(-\frac{N^2}{2}\mathcal{G}_j[\rho] + N\mathcal{T}_j[\rho]\right), \tag{123}$$

where

$$\mathcal{G}_j[\rho] = \frac{1}{2}\int_{\mathcal{R}_j} \lambda^2 \rho(\lambda) d\lambda - \int_{\mathcal{R}_j}\int_{\mathcal{R}_j} \rho(\lambda)\rho(\lambda') \ln(|\lambda - \lambda'|) d\lambda d\lambda', \tag{124}$$

$$\mathcal{T}_j[\rho] = \int_{\mathcal{R}_j} \rho(\lambda) \ln(|\lambda + s|) d\lambda - \int_{\mathcal{R}_j} \rho(\lambda) \ln \rho(\lambda) d\lambda. \tag{125}$$

In the limit $N \to \infty$ in (123) the main contribution to the functional integral in (123) comes obviously from the value of ρ which minimises the functional $\mathcal{G}_j$. The stationary condition is found in the standard variational procedure after incorporating the normalisation condition $\int_{\mathcal{R}_j} \rho(\lambda) d\lambda = 1$ via a Lagrange multiplier. The resulting integral equation reads:

$$\frac{\lambda^2}{4} + C = \int_{\mathcal{R}_j} \rho(\lambda') \ln(|\lambda - \lambda'|) d\lambda' \quad \text{for } \lambda \in \mathcal{R}_j. \tag{126}$$

Differentiating this equation with respect to λ gives

$$\frac{\lambda}{2} = \int_{\mathcal{R}_j} \frac{\rho(\lambda') d\lambda'}{\lambda - \lambda'} \quad \text{for } \lambda \in \mathcal{R}_j, \tag{127}$$

where the integral must now be understood as a Cauchy principal value.

Solution of singular integral equations of the type (127) is discussed extensively in [61]. The necessary inversion formula essentially depends on whether the solution is required to be bounded at each end. In general, given a function $g(\lambda)$ for $\lambda \in (a, b)$ and the integral equation

$$g(\lambda) = \int_a^b \frac{f(\lambda') d\lambda'}{\lambda - \lambda'}$$

there is a unique solution $f(\lambda)$ which remains bounded at the endpoints a and b provided that holds

$$\int_a^b \frac{g(\lambda') d\lambda'}{\sqrt{(b - \lambda')(\lambda' - a)}} = 0.$$

In our case, the above condition implies

$$0 = \int_a^b \frac{\lambda' d\lambda'}{\sqrt{(b - \lambda')(\lambda' - a)}} = \frac{\pi(a + b)}{2}. \tag{128}$$

Hence, such a solution only exists in the case $a = -b \equiv L/2$, and in the latter case is given by the inversion formula

$$\rho(\lambda) = \frac{\sqrt{L^2/4 - \lambda^2}}{2\pi^2} \int_{-L/2}^{L/2} \frac{\lambda' d\lambda'}{\sqrt{L^2/4 - \lambda'^2}(\lambda' - \lambda)}, \tag{129}$$

$$= \frac{\sqrt{L^2/4-\lambda^2}}{2\pi^2}\left[\int_{-L/2}^{L/2}\frac{d\lambda'}{\sqrt{L^2/4-\lambda'^2}} + \lambda\int_{-L/2}^{L/2}\frac{d\lambda'}{\sqrt{L^2/4-\lambda'^2}(\lambda'-\lambda)}\right], \quad (130)$$

$$= \frac{\sqrt{L^2/4-\lambda^2}}{2\pi}, \quad (131)$$

where in the final line we have used the identity

$$\int_a^b \frac{dx}{\sqrt{(b-x)(x-a)}(x-z)} = 0 \quad \text{for } z \in (a,b). \quad (132)$$

The normalization condition fixes L, and the resulting eigenvalue density is given by

$$\rho_{sc}(\lambda) = \frac{1}{2\pi}\sqrt{4-\lambda^2}, \quad \text{for } \lambda \in [-2,2] \quad (133)$$

and zero otherwise, which is just the well-known Wigner semi-circle law. In such a case the only dependence on the variable s in the exponent of $\mathcal{D}_s(s)$ comes from the first term $\int_{-2}^{2}\rho_{sc}(\lambda)\ln(|\lambda+s|)d\lambda$ in (125) which leads to (37) as described in the text.

At the same time, when evaluating $\mathcal{D}_m(s)$ we have an additional constraint on the density ρ, as the latter must vanish for $\lambda < -s$. Obviously, the solution (133) can satisfy such a constraint only as long as $s > 2$, and has to be modified in the opposite case $s > 2$. As shown in [61] there always exists a solution of this type of integral equations which is bounded only at one end of the integration range. The solution which remains bounded at the upper end of the integration range is given by

$$\rho_{DM}(\lambda) = \frac{1}{2\pi^2}\sqrt{\frac{L-s-\lambda}{\lambda+s}}\int_{-s}^{L-s}\sqrt{\frac{\lambda'+s}{L-s-\lambda'}}\frac{\lambda' d\lambda'}{\lambda-\lambda'}, \quad (134)$$

where L is a constant to be determined by the normalization. The integral was further evaluated by Dean and Majumdar and the resulting density is given, for our choice of the GOE measure, by

$$\rho_{DM}(\lambda) = \begin{cases} \dfrac{1}{4\pi}\sqrt{\dfrac{L-\lambda-s}{\lambda+s}}[L+2\lambda], & \text{if } 0 \le \lambda+s \le L, \\ 0, & \text{otherwise,} \end{cases} \quad (135)$$

where this time

$$L = \frac{2}{3}\left(s+\sqrt{s^2+12}\right). \quad (136)$$

Note that $s \to 2$ implies $L \to 4$ and the Dean–Majumdar density (135) reverts in this limit to the Wigner semicircular law (133).

Inserting the above function ρ_{DM} in the definitions of $\mathcal{G}_m$ and $\mathcal{T}_m$, see (124), (125) and performing the integrations with help of MATHEMATICA yields

$$\begin{aligned}\mathcal{G}_m[\rho_{DM}] &\equiv G_{\min}(s) \\ &= \frac{1}{216}\left(72s^2 - s^4 - 30s\sqrt{12+s^2} - s^3\sqrt{12+s^2} + 54(3+4\ln 6)\right) \\ &\quad - \ln\left(s+\sqrt{s^2+12}\right), \end{aligned} \quad (137)$$

which is our expression (69), as well as a formula for $\mathcal{T}_m[\rho_{DM}]$. The latter formula is however irrelevant for finding the complexity of minima, as our analysis reveals that the complexity is determined by value $s \to 2$ when obviously $\mathcal{T}_m[\rho_{DM}] \to \mathcal{T}_m[\rho_{sc}]$.

Appendix 4: Analysis of the Replica-Symmetric Solution for a Spherically-Symmetric Confining Potential

The calculation of the de-Almeida-Thouless condition in the general energy surface of the form (11) requires only very minor modifications in comparison with the case of a parabolic confinement (6) considered in detail in [15]. Applying the procedure of Ref. [15] yields the following exact expression for the averaged replicated partition function:

$$\langle Z_\beta^n \rangle = \mathcal{C}_{N,n} N^{Nn/2} e^{\frac{\beta^2}{2} Nnf(0)} \int_{Q>0} (\det Q)^{-(n+1)/2} e^{-\beta N \Phi_n(Q)} dQ, \tag{138}$$

where

$$\Phi_n(Q) = \sum_{a=1}^{n} U\left(\frac{q_{aa}}{2}\right) - \frac{1}{2\beta} \ln(\det Q) - \beta \sum_{a<b} f\left[\frac{1}{2}(q_{aa} + q_{bb}) - q_{ab}\right], \tag{139}$$

where $C_{N,n}$ is a known numerical constant and N is assumed to satisfy the constraint $N > n$.

The form of the integrand in (138) is precisely one required for the possibility of evaluating the replicated partition function in the limit $N \to \infty$ by the Laplace ("saddle-point") method. The free energy is then given by

$$F_\infty = \lim_{N\to\infty} \frac{1}{N} \langle F \rangle = -\frac{T}{2} \ln(2\pi e) - \frac{1}{2T} f(0) + \lim_{n\to 0} \frac{1}{n} \Phi_n(Q), \tag{140}$$

where the entries of the matrix Q are chosen to satisfy the stationarity conditions: $\frac{\partial \Phi_n(Q)}{\partial q_{ab}} = 0$ for $a \le b$. This yields, in general, the system of $n(n+1)/2$ equations:

$$\frac{1}{2}\mu(q_{aa}) - \frac{1}{\beta}[Q^{-1}]_{aa} - \beta \sum_{b(\neq a)}^{n} f'\left[\frac{1}{2}(q_{aa} + q_{bb}) - q_{ab}\right] = 0, \quad a = 1, 2, \ldots, n \tag{141}$$

and

$$-\frac{1}{\beta}[Q^{-1}]_{ab} + \beta f'\left[\frac{1}{2}(q_{aa} + q_{bb}) - q_{ab}\right] = 0, \quad a \neq b, \tag{142}$$

where $f'(x)$ stands for the derivative df/dx and we used the convention (12).

The Replica Symmetric Ansatz amounts to searching for a solution to (141), (142) within subspace of matrices $Q = Q_{RS} > 0$ such that $q_{aa} = q_d$, for any $a = 1, \ldots, n$, and $q_{a<b} = q_0$. The system of equations is easy to solve and to obtain in the replica limit $n \to 0$ the following relations:

$$q_d = \frac{T}{\mu_d} - \frac{1}{\mu_d^2} f'\left(\frac{T}{\mu_d}\right), \quad q_0 = -\frac{1}{\mu_d^2} f'\left(\frac{T}{\mu_d}\right), \tag{143}$$

where we denoted for brevity $\mu_d \equiv \mu(q_d)$.

The stability analysis of this solution amounts to expanding the function $\Phi_n(Q)$ in (139) around the extremum point up $Q = Q_{RS}$ to the second order in deviations: $\Phi = \Phi_{SP} +$

$\delta\Phi + \frac{1}{2}\delta^2\Phi$. The stationarity condition amounts to $\delta\Phi = 0$ yielding the system (141), (142). The term $\delta^2\Phi$ is a quadratic form in independent fluctuation variables $\delta q_{ab}, a \leq b$ and can be generally written as $\delta^2\Phi = \sum_{(ab),(cd)} \delta q_{(ab)} G_{(ab),(cd)}\, \delta q_{(cd)}$. As usual the stable extremum corresponds to the positive definite quadratic form, and along the critical line the quadratic form becomes semi-definite. Checking positive definiteness of $\delta^2\Phi$ amounts to finding the (generalized) eigenvalues Λ of the matrix $G_{(ab),(cd)} = \frac{\partial^2\Phi}{\partial q_{ab}\partial q_{cd}}$. It is easy to see that in our case the quadratic form can be written as

$$\delta^2\Phi = \frac{\mu'(q_d)}{2}\sum_a^n (\delta q_{ab})^2 + \frac{T}{2}\mathrm{Tr}[\delta Q(Q_{RS})^{-1}\delta Q(Q_{RS})^{-1}] - \frac{1}{T}\sum_{ab} f''(D_{ab})\delta D_{ab}^2, \quad (144)$$

where we introduced short-hand notations $\delta D_{ab} = \frac{1}{2}(\delta q_{aa} + \delta q_{bb} - 2\delta q_{ab})$ and $D_{ab} = q_d - q_{ab}$. The corresponding eigen-equations for generalized $n(n+1)/2$ component eigenvectors $\eta_{(ab)}$ with $a < b$ can be written straightforwardly by repeating the analysis of [15] and are given for $T > 0$ by:

$$\sum_{cd}(Q_{RS}^{-1})_{ac}\eta_{(cd)}(Q_{RS}^{-1})_{db} + \frac{1}{T^2}f''(D_{ab})(\delta D_{ab}) + \delta_{ab}\left[\frac{2}{T}\mu'(q_d) - \sum_c f''(D_{ac})(\delta D_{ac})\right] = \Lambda^*\eta_{(ab)}. \quad (145)$$

As the entries of the matrix Q_{RS} are given by $q_{ab} = q_0 + (q_d - q_0)\delta_{ab}$, its inverse Q^{-1} has the same form $(Q^{-1})_{ab} = p_0 + (p_d - p_0)\delta_{ab}$, with p_0 and $p_d - p_0$ are given in the limit $n \to 0$ by

$$p_0 = -\frac{q_0}{(q_d - q_0)^2} = \frac{1}{T^2}f'\left(\frac{T}{\mu_d}\right), \quad p_d - p_0 = \frac{1}{q_d - q_0} = \frac{\mu_d}{T}. \quad (146)$$

Now we can follow faithfully the lines of the classical work by De Almeida and Thouless [3] and to provide an explicit construction of the families of eigenvectors with components $\eta_{(ab)}$ of different symmetry. For the model with parabolic confinement this construction was discussed in [15] and goes through here without any modification. There are three different families of eigenvectors, first two yielding only eigenvalues λ^* with positive real part, hence stable. The dangerous third family of eigenvectors is that satisfying the constraints:

$$\eta_{(aa)}^{(III)} = 0, \quad \forall a; \qquad \sum_d \eta_{(ad)}^{(III)} = 0, \quad \forall a. \quad (147)$$

The equations (145) are then reduced to a single equation

$$\left[(p_d - p_0)^2 - \frac{1}{T^2}f''(q_d - q_0)\right]\eta_{(ab)}^{(III)} = \Lambda^*\eta_{(ab)}^{(III)}, \quad \forall a \neq b. \quad (148)$$

and substituting here (146) we find that the replica symmetric solution is stable as long as

$$\mu_d^2 \geq f''\left(\frac{T}{\mu_d}\right) \quad (149)$$

with the equality $\mu_d^2 = f''(\frac{T}{\mu_d})$ providing the condition of the replica symmetry breaking transition. Solving the latter together with (143) is easy after introducing the auxiliary variable $\tau = \frac{T}{\mu_d}$. This gives finally the transition temperature line in the form

$$T_{AT} = \tau\sqrt{f''(\tau)}, \tag{150}$$

where τ satisfies the equation

$$\mu\left(\tau - \frac{f'(\tau)}{f''(\tau)}\right) = \sqrt{f''(\tau)}. \tag{151}$$

In particular, the transition at zero temperature $T \to 0$ requires $\tau \to 0$, hence the replica symmetric solution at zero temperature is stable as long as the inequality

$$\mu\left(-\frac{f'(0)}{f''(0)}\right) \geq \sqrt{f''(0)}. \tag{152}$$

is satisfied, with the equality sign standing for the transition condition to the region with broken replica symmetry.

Appendix 5: Complexity of Stationary Points with a Given Index

In this appendix we outline the calculation of complexity for the stationary points with a given index $\mathcal{I}(\hat{H})$ (the number of negative eigenvalues of the Hessian $\hat{H}$) for our model. We closely follow the method developed by Bray and Dean in [54], which allows one to perform the calculation for the extensive values of the index scaled with N as $\mathcal{I} = \alpha N$, where $0 \leq \alpha < 1$. For simplicity we consider only the case of parabolic confinement, with constant $\mu(z) = \mu$.

Repeating the same steps as in [51], the expected value of the number $\mathcal{N}(\mu, \alpha)$ of stationary points with a given value of α can be straightforwardly shown to be given by (cf. (9)):

$$\langle \mathcal{N}(\mu,\alpha)\rangle = \frac{1}{s^N}\frac{1}{2^N}\sqrt{\frac{2}{N+2}}\left(\frac{N}{\pi}\right)^{N(N+1)/4} \times \int d\hat{H}|\det(sI_N + \hat{H})|\delta(\mathcal{I}(sI_N + \hat{H}) - N\alpha)e^{-\frac{N}{4}[\mathrm{Tr}\hat{H}^2 - \frac{1}{N+2}(\mathrm{Tr}\hat{H})^2]}, \tag{153}$$

where we denoted $s = \mu/\sqrt{f''(0)} \equiv \mu/\mu_{cr}$.

The integration over the real, symmetric matrix $\hat{H}$ is in the standard way reduced to an integral over its eigenvalues λ_i, and further to the functional integral over the mean eigenvalue density, (114), see Appendix 2. This gives:

$$\langle \mathcal{N}(\mu,\alpha)\rangle = \frac{1}{s^N}\frac{\Psi(\mu,\alpha)}{D}, \tag{154}$$

where

$$\Psi(\mu,\alpha) = \int \mathcal{D}[\rho]\exp(-N^2\mathcal{S}_2[\rho] + N\mathcal{S}_1[\rho, s]) \times \delta\left(\int_{-\infty}^{\infty}\rho(\lambda)d\lambda - 1\right)\delta\left(\int_{-\infty}^{-s}\rho(\lambda)d\lambda - \alpha\right). \tag{155}$$

The expressions appearing in the exponential in (155) are given by

$$\mathcal{S}_2[\rho] = \frac{1}{4}\int_{-\infty}^{\infty}\rho(\lambda)\lambda^2 d\lambda - \frac{1}{4}\left(\int_{-\infty}^{\infty}\rho(\lambda)\lambda d\lambda\right)^2 - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\rho(\lambda)\rho(\lambda')\ln(|\lambda-\lambda'|)d\lambda d\lambda', \tag{156}$$

and

$$\mathcal{S}_1[\rho, s] = \int_{-\infty}^{\infty}\rho(\lambda)\ln(|s+\lambda|)d\lambda - \int_{-\infty}^{\infty}\rho(\lambda)\ln\rho(\lambda)d\lambda - \frac{N}{2(N+2)}\left(\int_{-\infty}^{\infty}\rho(\lambda)\lambda d\lambda\right)^2. \tag{157}$$

The denominator D appearing in (154) is a normalization factor which can also be represented as a functional integral and is given by

$$D = \int \mathcal{D}[\rho]\exp(-N^2\mathcal{S}_2[\rho] + N\mathcal{S}_1'[\rho])\delta\left(\int_{-\infty}^{\infty}\rho(\lambda)d\lambda - 1\right), \tag{158}$$

where

$$\mathcal{S}_1'[\rho] = -\int_{-\infty}^{\infty}\rho(\lambda)\ln\rho(\lambda)d\lambda - \frac{N}{2(N+2)}\left(\int_{-\infty}^{\infty}\rho(\lambda)\lambda d\lambda\right)^2. \tag{159}$$

As $N \to \infty$, the main contribution to the functional integral in (155) comes from the function ρ which minimizes $\mathcal{S}_2[\rho]$. By using the definition of the mean eigenvalue,

$$\bar{\lambda} := \int_{-\infty}^{\infty}\rho(\lambda)\lambda d\lambda, \tag{160}$$

and further introducing the function f defined as

$$f(x) = \rho(x + \bar{\lambda}), \tag{161}$$

we rewrite $\mathcal{S}_2[\rho]$ in the following form:

$$\mathcal{S}_2[f] = \frac{1}{4}\int_{-\infty}^{\infty}f(x)x^2 dx - \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)f(x')\ln(|x-x'|)dxdx'. \tag{162}$$

We now need to find the function f which minimizes (162) subject to the constraint $\int f(x)dx = 1$. A simple variational calculation (cf. Appendix 3) shows that the minimizer is the usual Wigner semi-circle density, given by

$$f(x) = \frac{\sqrt{4-x^2}}{2\pi}. \tag{163}$$

The value of $\bar{\lambda}$ can now be fixed using the restriction on the index given by the second δ-functional factor in (161). This leads to

$$\alpha = \frac{2}{\pi}\int_{-1}^{-h/2}\sqrt{1-x^2}dx, \tag{164}$$

where $h = s + \bar{\lambda}$. Each value of $\alpha \in [0, 1]$ then corresponds to a unique value of $h \in [-2, 2]$.

We can apply the same variational procedure to the normalization factor D. The only difference is that the functional integration in the denominator does not contain the restriction on index, which results in second term in $\mathcal{S}_1'$ minimized by the value $\overline{\lambda} = 0$. On the other hand, the terms containing $\mathcal{S}_2$ appear in both D and Ψ, and therefore cancel from their ratio in (154) when we apply the asymptotic evaluation of the integrals in the limit $N \to \infty$. Likewise, the term of the form $\int \rho(\lambda) \log \rho(\lambda) d\lambda$ will cancel between the numerator and the denominator. Consequently, the relation (154) is reduced asymptotically to

$$\left\langle \mathcal{N}(\mu, \alpha) \right\rangle \sim \frac{1}{s^N} \exp\left(N \left[\int_{-2}^{2} f(x) \ln(|x + h|) - \frac{1}{2} \bar{\lambda}^2 \right] \right), \tag{165}$$

where $\overline{\lambda} = h - s$, and h is related to α by (164). For the semicircular form of $f(x)$ the integral above can be explicitly calculated:

$$\frac{1}{2\pi} \int_{-2}^{2} \sqrt{4 - x^2} \ln|x + h| \, dx = -\frac{1}{2} + \frac{h^2}{4}. \tag{166}$$

Substituting this result to (165) yields the final form of the complexity corresponding to a given index:

$$\Sigma(s, \alpha) = -\frac{1}{2} + \frac{h^2}{4} - \frac{(h - s)^2}{2} - \ln s. \tag{167}$$

The analysis of this expression is convenient to perform separately for $\alpha = 0$ and $\alpha > 0$.

1. The relation (164) for $\alpha = 0$ implies $h = 2$. Inserting this in (167) gives

$$\Sigma(s, 0) = \frac{1}{2} - \frac{(2 - s)^2}{2} - \log s \,. \tag{168}$$

This expression is zero at $s = 1$, and taking the derivative with respect to s gives

$$\frac{\partial}{\partial s} \Sigma(s, 0) = 2 - \left(s + \frac{1}{s} \right) \leq 0 \tag{169}$$

with equality achieved only for $s = 1$. We see that $\Sigma(s, 0)$ is decreasing with s and hence is positive for $0 < s < 1$. Also $\frac{\partial^2}{\partial s^2} \Sigma(s, 0) = \frac{1}{s^2} - 1$, $\frac{\partial^3}{\partial s^3} \Sigma(s, 0) = -\frac{2}{s^3}$. Thus, the first non-vanishing derivative at $s = 1$ is the third derivative, implying that the complexity of stationary points with $\alpha = 0$ must vanish cubically as $s \to 1$. This agrees with the general analysis of complexity of minima performed earlier in the paper.

2. Equation (164) implies $h < 2$ for $\alpha > 0$. Differentiating (167) with respect to s gives

$$\frac{\partial}{\partial s} \Sigma(s, \alpha) = h - \left(s + \frac{1}{s} \right) < 0. \tag{170}$$

Thus, $\Sigma(s, \alpha)$ is strictly decreasing as a function of s, and vanish *linearly* at some critical point s_{cr} which can be easily found numerically as a function of α. Observing that

$$\Sigma(1, \alpha) = -\frac{(h - 2)^2}{4} < 0 \tag{171}$$

we conclude that the critical value $s_{cr}(\alpha)$ must satisfy $s_{cr}(\alpha) < 1$ for any $\alpha > 0$. In other words, the complexity of stationary points with any extensive index $\mathcal{I} = O(N)$ is still

negative at the point of ergodicity breaking $s = 1$ (i.e. $\mu = \mu_{cr} = \sqrt{f''(0)}$ in the original notations), and starts to be positive already inside the phase with broken ergodicity: $\mu_{cr}(\alpha > 0) < \sqrt{f''(0)}$.

If we considered the annealed average of the total number of stationary points, rather than those with a fixed index, this would be equivalent to integrating $\langle \mathcal{N}(\mu, \alpha) \rangle$ over all values of α. In the limit $N \gg 1$ such an integral will obviously be dominated by the value of α which maximizes $\Sigma(s, \alpha)$. As $d\alpha/dh \neq 0$ for $|h| < 2$ the maximum occurs when

$$0 = \frac{\partial}{\partial h} \Sigma(s, \alpha) = s - \frac{h}{2}, \tag{172}$$

so that $h = 2s$. At this point of maximum

$$\Sigma_{tot}(s) = \frac{s^2 - 1}{2} - \log s. \tag{173}$$

This shows that complexity of all stationary points tends to zero as $s \to 1$, and by taking derivatives we find that at $s = 1$ the first derivative vanishes but the second one is non-zero. Consequently, the (annealed) complexity related to the total number of stationary points vanishes as $(1 - s)^2$ as $s \to 1$, as indeed was found in [51] by a different method.

Moreover, the last result can be used to show that the critical value $s_{cr}(\alpha)$—defined as s at which the complexity with a given α vanishes—must monotonously decrease with α increasing. First observe that in fact we have shown that $\Sigma(h/2, \alpha) > 0$ which implies $s_{cr}(\alpha) > h/2$. The value $s_{cr}(h)$ by definition solves $\Sigma(s_{cr}(h), h) = 0$. As h decreases with α increasing, it is enough to show that $ds_{cr}/dh > 0$. Taking the derivative with respect to h and rearranging gives

$$\frac{ds_{cr}}{dh} = -\left(\frac{\partial \Sigma / \partial h}{\partial \Sigma / \partial s} \right) = \frac{s_{cr} - h/2}{s_{cr} + s_{cr}^{-1} - h} > 0 \tag{174}$$

as required.

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